One may further apply the relation [see, for example, Ref. (4)] $D_-(k) = D_+(-k)$, which we gave for energies much larger than the rest mass of the meson. Eq. (9) then takes the following form:

$$f_5 = (D_+(k) - D_+(-k)) / \sqrt{2}. \tag{10}$$

It is easily seen from the dispersion relations that link the real part of the forward scattering amplitude with the total scattering cross section,[4] that within given limits, the total cross sections at high energies $D_\pm(k)$ cannot increase faster than the first power of k. Thus, even if $D_+(k) = \alpha k$, where α is a constant, $D_-(k) = D_+(-k) = -\alpha k$, and therefore, $f_5(k) = \sqrt{2\alpha k} = \sqrt{2}\, D_+(k)$. Owing to the fact that at large energies, from Eq. (6), $|f_5| \ll |f_1|$, and $f_1 = D_+ + iA_+$, the following inequality holds:

$$|D_+(k)| \ll |A_+(k)|. \tag{11}$$

Note that according to Eq. (8), $A_+(k)$ increases with energy as the first power of k as long as σ_0 does not vanish, thus if $D_+(k)$ increases more slowly, Eq. (11) is fulfilled all the more.

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INVERSE DISPERSION RELATIONS

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"Inverse" dispersion relations express the imaginary part of a scattering amplitude as a Cauchy integral of the real part. The integral over the unphysical region is evaluated by making use of ordinary "direct" dispersion relations. The forward scattering amplitudes for charged pions on nucleons are discussed in detail. The inverse dispersion relations for this case are written out explicitly in a form in which no unobservable quantities appear.

STARTING from quantum field theory, various authors[1,2] have derived dispersion relations for the scattering of pions and photons on nucleons and for the photoproduction of pions. These relations express the real part $D(E)$ of the amplitude as a Cauchy integral of the imaginary part $A(E)$, thus

$$D(E) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{A(E')}{E' - E} dE'. \tag{1}$$

The negative part of the energy range can be eliminated by using symmetry properties, and for forward scattering the part of the integral arising from unphysical positive energies can be computed explicitly. This reflects the fact that in the unphysical region the function $A(E)$ can be represented as a sum of δ-functions. The integral over the unphysical range then depends only on the coupling constants of the relevant interactions.

The dispersion relations for pion-nucleon scattering have been tested experimentally, and satisfactory agreement between theory and experiment has been found, over the range of energies now accessible.[3] A test of dispersion relations at higher energies would be of great interest. The derivations of dispersion relations all assume the validity of microscopic causality,[2,4] and if there is a failure of microscopic causality the relations should be altered in a definite way. However, dispersion relations of the form (1) may be unsuited for such a test. They will be inconvenient if at high energies the real part of the scattering amplitude becomes small compared with the imaginary part, and already there are some indications[5] that this will happen. If so, the relations equate a small quantity on the left side of Eq. (1) to an integral of a large quantity of variable sign on the right side. In this case, the already considerable experimental errors in the determination of A will produce an enormous overall effect. As a result, the experimental test of dispersion relations of the form of Eq. (1) at high energy may be very difficult, and the experimental observation of a fundamental length may be hindered.

If these difficulties prove to be real, it will be much more convenient to use inverse dispersion relations of the form

$$A(E) = -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{D(E')}{E'-E} dE', \tag{2}$$

which do not suffer from the same disadvantages. The problem then arises, that in order to use Eq. (2) we must determine the function $D(E)$ in the unphysical range of positive energies. This problem has not yet been solved, and Eq. (2) has consequently not been studied.

We can overcome the difficulty as follows. Bogoliubov[4] has shown that the direct dispersion relation (1) holds even when E lies in the unphysical region. The function D in the integrand of Eq. (2) can therefore be replaced in the unphysical positive-energy region by the expression (1), and all unobservable quantities are thereby eliminated from the equation. The double integral which results from this substitution can be reduced to a single integral by interchanging the order of integration.

Consider the amplitude for forward pion-nucleon scattering in the laboratory system,

$$f(E) = \delta_{\rho\rho'} f_1^1 + 2i(\sigma[\mathbf{q}\times\mathbf{q}'])\,\delta_{\rho\rho'} f_2^1 + \frac{[\tau_{\rho'}, \tau_\rho]}{2} f_2^1 + i(\sigma[\mathbf{q}\times\mathbf{q}'])\,[\tau_{\rho'}, \tau_\rho] f_2^2 \tag{3}$$

The scalar coefficients f_j^i satisfy the following inverse dispersion relations

$$\begin{aligned} A_i^i(E) = {} & \frac{D_i^i(E) - D_i^i(\mu)}{\pi} \ln\frac{E+\mu}{E-\mu} - \frac{2E(E^2-\mu^2)}{\pi} \int_\mu^\infty \frac{dE'}{(E'^2-E^2)(E'^2-\mu^2)} \left\{ D_i^i(E') - D_i^i(\mu) + \frac{A_i^i(E')}{\pi} \ln\frac{E'+\mu}{E'-\mu} \right\} \\ & - C_i \frac{f^2}{\pi} \frac{2E(E^2-\mu^2)}{E^2-(\mu^2/2M)^2} \ln\left(\frac{1+\mu/2M}{1-\mu/2M}\right) \Big/ [\mu^2 - (\mu^2/2M)^2] \quad (i=1,2); \end{aligned} \tag{4}$$

$$\begin{aligned} A_i^k(E) = {} & -\frac{2}{\pi}(E^2-\mu^2) \int_\mu^\infty \frac{dE'}{(E'^2-E^2)(E'^2-\mu^2)} \left\{ E' D_i^k(E') - \mu D_i^k(\mu) + \frac{A_i^k(E')}{\pi} \ln\frac{E'+\mu}{E'-\mu} \right\} \\ & + \left(D_i^k(E) - \frac{\mu}{E} D_i^k(E) \right) \frac{1}{\pi} \ln\frac{E+\mu}{E-\mu} - C_i \frac{f^2}{\pi} \frac{\mu^2}{M} \frac{E^2-\mu^2}{E^2-(\mu^2/2M)^2} \ln\left(\frac{1+\mu/2M}{1-\mu/2M}\right) \Big/ [\mu^2 - (\mu^2/2M)^2] \\ & (i \neq k;\ i,\ k = 1,2). \end{aligned} \tag{5}$$

Here

$$C_1 = 1, \quad C_2 = -1/\mu^2. \tag{6}$$

In the usual way we derive from Eq. (4) and (5) the more interesting dispersion relations for the scattering of charged pions by protons,

$$\begin{aligned} A_+(E) = {} & -\frac{E^2-\mu^2}{\pi} \int_\mu^\infty \frac{dE'}{E'^2-\mu^2} \mathrm{P} \frac{1}{E'-E} \left\{ D_+(E') - D_+(\mu) + \frac{A_+(E')}{\pi} \ln\frac{E'+\mu}{E'-\mu} \right\} \\ & + \frac{E^2-\mu^2}{\pi} \int_\mu^\infty \frac{dE'}{E'^2-\mu^2} \frac{1}{E'+E} \left\{ D_-(E') - D_-(\mu) + \frac{A_-(E')}{\pi} \ln\frac{E'+\mu}{E'-\mu} \right\} + \frac{1}{\pi} \left\{ D_+(E) - \frac{E+\mu}{2E} D_+(\mu) - \frac{E-\mu}{2E} D_-(\mu) \right\} \ln\frac{E+\mu}{E-\mu} \\ & + \frac{2f^2(E^2-\mu^2)}{\pi(E-\mu^2/2M)} \ln\left(\frac{\mu-\mu^2/2M}{\mu+\mu^2/2M}\right) \Big/ [\mu^2 - (\mu^2/2M)^2]; \end{aligned} \tag{7}$$

$$\begin{aligned} A_-(E) = {} & -\frac{(E^2-\mu^2)}{\pi} \int_\mu^\infty \frac{dE'}{E'^2-\mu^2} \mathrm{P} \frac{1}{E'-E} \left\{ D_-(E') - D_-(\mu) + \frac{A_-(E')}{\pi} \ln\frac{E'+\mu}{E'-\mu} \right\} + \frac{(E^2-\mu^2)}{\pi} \int_\mu^\infty \frac{dE'}{E'^2-\mu^2} \frac{1}{E'+E} \\ & \times \left\{ D_+(E') - D_+(\mu) + \frac{A_+(E')}{\pi} \ln\frac{E'+\mu}{E'-\mu} \right\} + \frac{1}{\pi} \left\{ D_-(E) - \frac{E+\mu}{2E} D_-(\mu) - \frac{E-\mu}{2E} D_+(\mu) \right\} \ln\frac{E+\mu}{E-\mu} \\ & + \frac{2f^2}{\pi} \left(\frac{E^2-\mu^2}{E+\mu^2/2M} \right) \ln\left(\frac{1-\mu/2M}{1+\mu/2M}\right) \Big/ [\mu^2 - (\mu^2/2M)^2]. \end{aligned} \tag{8}$$

The "large" function A appears on the right side of these relations. It might seem that the problem of obtaining relations in which only the small function appears in the integrals has not been completely solved. But in fact the "large" functions A occur in the integrals multiplied by a small logarithm, which drastically reduces their effect. Thus, supposing that the total charged pion-proton cross-sections at high energy are constant and (for example) equal to $3 \cdot 10^{-26}\,\text{cm}^2$, but that the functions D_+ and D_- are constant at high energy and approximately equal to $(0 \cdot 3/\mu)$, we find that the contribution of the imaginary part to the integral is five times smaller than that of the real part. These numbers are in agreement with Sternheimer's estimates.[5]

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