INVESTIGATION OF THE EQUILIBRIUM FORM OF ATOMIC NUCLEI

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It is shown that the experimentally observed abrupt transition from the spherical equilibrium form of atomic nuclei to the ellipsoidal form is explained by the influence of pair interactions of nucleons. If the magnitude of the difference between the pair energies in the ellipsoidal and spherical states is estimated from experimental data on the nuclear binding energies, then the theoretical regions of prolate nuclei coincide with those observed in experiment.

INTRODUCTION

THE concept of the independent motion of nucleons in the mean effective field of the nucleus is basic to contemporary nuclear models. As a first approximation, a spherically symmetric field was assumed,[1] which made it possible to explain the existence of nuclear shells, with consideration of strong spin-orbit coupling. The scheme of bound states in a spherical nucleus, which we call the Mayer scheme, allowed us to understand a number of regularities observed in the spectra of single particle excitations of the nuclei.

Along with single-particle excitations in atomic nuclei, collective excitations, which possess a rotational or vibrational character have been discovered. Nuclei with a rotational spectrum of collective motion have a prolate equilibrium shape, which also follows from the data on quadrupole moments. The spins and parities of single particle states in these nuclei do not satisfy the Mayer scheme. Therefore it was hypothesized that the effective field in these nuclei does not have spherical symmetry.[2,3] In this case, the energies of the bound states of the nucleons are functions of the deformation (by deformation of the nucleus, we mean the departure of the nuclear field $U(\mathbf{r})$ from spherically symmetric form), and the nucleonic configuration, together with the equilibrium deformation of the given nucleus, is determined from the condition of a minimum in its total energy.[4]

The field acting on the given nucleon is the average of its interaction with all the remaining nucleons, and therefore the symmetry of the potential $U(\mathbf{r})$ is determined by the spatial distribution of the nucleons in the nucleus. Inasmuch as the latter is isotropic for filled shells, then, strictly speaking, only the magic nuclei can satisfy the Mayer scheme. It can be expected that the remaining nuclei will be deformed, and their single particle states will satisfy the scheme of bound states in a deformed nucleus, which we call the Nilsson scheme.

However, the Nilsson scheme agrees with experiment only in three well known regions of rotational excitation: (1) the rare earths in the range $_{60}\mathrm{Nd}^{150} - {}_{76}\mathrm{Os}^{190}$, (2) the heavy nuclei, beginning with $_{88}\mathrm{Ra}^{222}$, and (3) the light nuclei in the region of $_{12}\mathrm{Mg}^{24}$. For nuclei outside the stated intervals, there is no indication of the presence of a deformation, and their single particle states satisfy the usual Mayer scheme.

To clarify the existence of sharp limits within which the nuclei are prolate, the conditions of transition to the deformed equilibrium form are investigated in the present research. For this purpose, we first consider the establishment of equilibrium deformation in the independent particle approximation; the role of pair interaction in the process under investigation is then explained and, finally, the results are applied to finding the limits of prolateness of the nuclei; finally, comparison is made with experiment

1. EQUILIBRIUM DEFORMATION IN THE INDEPENDENT-PARTICLE APPROXIMATION

The Hamiltonian of a nucleon in a spherical nucleus has the form

$$\hat{H}_M = -\frac{\hbar^2}{2m}\Delta + V(r) + \frac{C}{r}\frac{dV}{dr}\hat{\mathbf{l}}\hat{\mathbf{s}}. \tag{1}$$

The solution of the Schrödinger equation with the Hamiltonian (1) gives the wave functions and the energies of the bound states or the nucleons in the spherical nucleus. The state $|N\ell j\rangle$ is characterized by a principal quantum number N, an orbital angular momentum ℓ and a total angular momentum j. For the proper choice of the spin-orbit coupling constant C, the energy spectrum of the system reproduces the experimentally observed order of levels in the spherical nucleus.[5] The nucleonic configuration of the nucleus is determined by the numbers of nucleons in the filled states, and the total energy of the nucleus in the independent particle approximation is the sum of the energy of the nucleons for a given configuration.

In a deformed nucleus, one can always find a surface

$$r'(\theta,\varphi)=r\Big[1+\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta,\varphi)\Big], \tag{2}$$

on which the potential of the nucleus $U(\mathbf{r})$ has a constant value. Denoting $U(\mathbf{r}')=V(\mathbf{r})$ and limiting ourselves to terms of second order in the expansion (2), we get

$$U(\mathbf{r})=V(r)+\frac{5}{8\pi}\beta^2r^2\frac{d^2V}{dr^2}-r\frac{dV}{dr}\sum_{\mu}\alpha_\mu Y_{2\mu}(\theta,\varphi), \tag{3}$$

where $\beta\sim\Delta R/R$ is the deformation parameter of the nucleus, and is related to α_μ by the expression $\beta^2=\sum_\mu|\alpha_\mu|^2$.

In the case of axially symmetric deformations, the Hamiltonian of the nucleon has the form

$$\hat{H}_N\hat{H}_0+\hat{H}_\beta=-\frac{\hbar^2}{2m}\Delta+V(r)+\frac{5}{8\pi}\beta^2r^2\frac{d^2V}{dr^2}+\frac{C}{r}\frac{dV}{dr}\,\hat{\mathbf{l}}\hat{\mathbf{s}}-r\frac{dV}{dr}\beta Y_{20};\quad \hat{H}_\beta=\frac{C}{r}\frac{dV}{dr}\,\hat{\mathbf{l}}\hat{\mathbf{s}}-r\frac{dV}{dr}\beta Y_{20}. \tag{4}$$

Solution of the Schrödinger equation with the Hamiltonian (4) gives the wave functions and the energies of the bound states in the deformed nucleus. The state $|\Omega\rangle$ is now characterized by the projection Ω of the total angular momentum on the axis of deformation (we choose this to be the z axis) and by its parity. For $\beta=0$, as is evident from (1) and (4), the schemes of Mayer and Nilsson coincide. Under the action of $\hat{H}_\beta$, each spherical state with given j is split into $(2j+1)/2$ states with $|\Omega|=1/2, 3/2, \ldots, j$. The states with $|\Omega|\ll j$ have a prolate orbit in the direction of the deformation axis and their completion contributes to the elongation of the nucleus. The states with $|\Omega|\approx j$ have an orbit which is perpendicular to the axis of deformation and upon its completion the nucleus is flattened. Therefore, for $\beta>0$, the levels are separated from the bottom upward in the order of increase of $|\Omega|$, while for $\beta<0$ the order of splitting is reversed.

Upon increase in the deformation, a crossing of levels takes place which arises as the result of the splitting of the spherical terms with different j. Therefore, for small deformations, the Nilsson scheme is strongly dependent on the order of the levels for $\beta=0$, and also on the form of the potential $V(\mathbf{r})$. We shall show that in the limiting case of large deformations, the Nilsson scheme depends slightly on these factors. For this purpose, we recall that the order of the levels for $\beta=0$ is determined on the basis of the magnitude of the spin-orbit coupling, and can change with variation of the constant C. In the case of large deformations, the spin-orbit coupling is greatly reduced, and the order of the levels is determined on the basis of the factor βY_{20}, by which the weak dependence of the Nilsson scheme on the form of the potential is also explained (cf. Refs. 2, 3). Actually, we can represent a state with given Ω in the form of a superposition of spherical states

$$|\Omega\rangle=\sum_{j=\Omega}^{N+1/2}C_j|Nlj\Omega\rangle,\qquad \sum_j|C_j|^2=1, \tag{5}$$

where all the states of the N-th shell give, at large deformations, almost identical contributions ($C_\Omega\approx C_{\Omega+1}\approx\ldots\approx C_{N+\frac{1}{2}}$). Then the matrix element of spin-orbit coupling is equal to

$$|\langle\Omega|2\hat{\mathbf{l}}\hat{\mathbf{s}}|\Omega\rangle|=\Big|\sum_j|C_j|^2\langle lj|2\hat{\mathbf{l}}\hat{\mathbf{s}}|lj\rangle\Big|=\Big|\sum_l\{|C_{l+1/2}|^2l-|C_{l-1/2}|^2(l+1)\}\Big|, \tag{6}$$

which is much smaller than the same matrix element in the spherical state [after exclusion of a few levels with $\Omega=N+\frac{1}{2}$, when only a single term enters into the expansion (5)]. Since, aside from (6), the relation

$$|\langle\Omega|\frac{C}{r}\frac{dV}{dr}\hat{\mathbf{l}}\hat{\mathbf{s}}|\Omega\rangle|\ll|\langle\Omega|r\frac{dV}{dr}\beta Y_{20}|\Omega\rangle|,$$

is satisfied for all Ω, the independence of the Nilsson scheme of the spin-orbit coupling for large deformations is thus shown.

The ground state of the nucleus is determined by the nucleonic configuration which corresponds to the smallest energy.

For nuclei with filled shells, the configurations in the Nilsson and Mayer schemes coincide, and the equilibrium deformation is equal to zero. In the presence of external nucleons, the Nilsson scheme gives the lower energy, since the energy of the lowest deformed state is less than the energy of the corresponding spherical terms. (For example, for positive deformations, the states with $|\Omega| \ll j$ are filled first; their energy for $\beta > 0$ is less than for $\beta = 0$.) The Coulomb energy, which is included in the total energy of the nucleus, and has the form

$$E_C = \frac{3}{5}\frac{Z^2e^2}{R}\left(1-\frac{4}{45}\varepsilon^2\right), \quad \varepsilon = \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta, \tag{7}$$

is also less in the deformed state. The difference $E_N - E_M = \Delta E(A)$ between the energies of the nucleus in the deformed and spherical states is always negative and can serve as a measure of the equilibrium deformation (for a filled shell, $\Delta E = 0$). With increase in the number of nucleons (or holes) outside a filled shell, $\Delta E(A)$ increases in absolute value, reaches a maximum when the shell is half filled, and then decreases to zero. The equilibrium deformation, correspondingly, changes continuously from nucleus to nucleus, and thus there is no jump in the equilibrium form.

2. CONSIDERATION OF PAIR INTERACTION OF NUCLEONS

Apart from the sum of the single particle energies of the nucleons and the Coulomb energy, the energy of the remaining interactions also enters into the total energy of the nucleus. The latter is introduced in the single particle model theory of shells, in order to take into account the experimentally observed spin dependence of the nuclear forces (the state of the system of independent particles is degenerate in the total spin).

We consider the contribution to the energy which gives the interaction of nucleons in the same quantum state, the so-called pair interaction. We shall consider the pair forces to be extremely short range and choose the operator of interaction in the form

$$\hat{W}(1.2) = -g\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{8}$$

Inasmuch as the pair interaction in the form (8) reproduces the empirical rule of spins, then it can be assumed that (8) correctly describes the qualitative regularities connected with the effect of pair forces, even if it does not pretend to yield quantitative agreement with experiment.

The energy of two monoenergetic nucleons in a spherical state with total angular momentum I is equal to[6]

$$w(j^2I) = {}^1/_2(2j+1+I)(2j-I)!\,I!^2(j-{}^1/_2+{}^1/_2I)!^2[(2j+I)!\,({}^1/_2I)!^4(j-{}^1/_2-{}^1/_2I)!^2]^{-1}F^0, \tag{9}$$

where j is the individual angular momentum of the nucleon,

$$F^0 = -\frac{g}{4\pi}\int_0^\infty R_{Nl}^4 r^2 dr,$$

and $R_{N\ell}$ is the radial part of the wave function. The state with momentum $I = 0$ corresponds to a minimum in the pair energy. For all j encountered in the periodic system,

$$w(j^22)/w(j^20) < {}^1/_4, \quad w(j^24)/w(j^20) < {}^1/_7, \quad w(j^26)/w(j^20) < {}^1/_{11}$$

etc.

We shall now show that the pair energy in the deformed state of the nucleus is always greater than the minimum energy in the spherical state. For this purpose, we expand the wave function in spherical states with definite total angular momentum [see Eqs. (5) — (7) in the Appendix]:

$$\Psi_\Omega(1,2)=\sum_{=0}^{2j-1} D^I_\Omega \Psi_{j^2I}(1,2), \qquad \sum_{I=0}^{2j-1} |D^I_\Omega|^2 = 1, \tag{10}$$

where

$$\Psi_\Omega(1,2)=\frac{1}{\sqrt{2}}[\Psi_{j,\Omega}(1)\Psi_{j,-\Omega}(2)-\Psi_{j,-\Omega}(1)\Psi_{j,\Omega}(2)], \quad \Psi_{j^2I}(1,2)=\sum_{\Omega=-j}^{j} C^{I0}_{\Omega,-\Omega}\Psi_{j,\Omega}(1)\Psi_{j,-\Omega}(2). \tag{11}$$

Then the pair energy in the deformed state of the nucleus is equal to

$$w_\Omega=\sum_{I=0}^{2j-1}|D^I_\Omega|^2 w(j^2I) > w(j^20)\left\{|D^0_\Omega|^2+\frac{1}{4}|D^2_\Omega|^2+\ldots\right\} > w(j^20). \tag{12}$$

The relation (12) is a consequence of the fact that the deformed state of the system is not an eigenstate of the total angular momentum.

In Table I we have listed the experimental values for the pair energy in the regions of heavy and light nuclei, obtained from data on the binding energies.[7,8] As can be seen, the pair energy decreases in absolute value in the transition to an elongated nucleus. In the region of heavy nuclei, the average value of the pair energy changes by 0.5 Mev in transition to elongated nuclei. In the region of light nuclei, the corresponding change amounts to 2 Mev. In the region of rare earths, the binding energies are not measured accurately, and cannot be used to determine the change in the pair energy in the transition. In this region, the energy ought to be somewhat higher than in the heavy nuclei, since the pair energy is an almost monotonically decreasing function of the mass number. For this region, we have taken the value 0.7 Mev, which is evidently close to the correct one.

TABLE I

Protons pair	Protons w, Mev	Neutrons pair	Neutrons w, Mev
Spherical nuclei		Spherical nuclei	
83—84	—1.373	127—128	—1.418
85—86	—1.689	129—130	—1.518
87—88	—1.784	131—132	—1.778
		133—134	—1.875
Prolate nuclei		135—136	—1.875
		137—138	—1.988
89—90	—1.068	Prolate nuclei	
91—92	—1.030		
93—94	—1.024		
95—96	—0.996	133—134	—1.359
97—98	—1.027	135—136	—1.268
99—100	—1.050	137—138	—1.389
Vicinity of Mg^{24}; nucleons		139—140	—1.340
Spherical nuclei		141—142	—1.186
		143—144	—1.074
9—10	—5.5	145—146	—1.132
		147—148	—1.107
Prolate nuclei		149—150	—1.092
		151—152	—1.100
11—12	—3.5		
13—14	—3.5		

The energy of pair interaction is equal to the sum of the pair energies in all pairs of nucleons. The difference $W_N - W_M - \Delta W(A)$ between the pair energies of the nucleus in the deformed and spherical states depends on the number of nucleons (or holes) outside a filled shell, but, in contrast to the difference $\Delta E(A)$ of single particle energies, they are always positive. If $\Delta E(A)$ is smaller than $\Delta W(A)$ in absolute magnitude, then the transition to the deformed equilibrium form is energetically forbidden. However, the nucleus has a spherical equilibrium form if the sum $\Delta E + \Delta W$ is positive.

Transition to the deformed equilibrium form is achieved upon filling those nucleonic levels in which the difference between the energies in the deformed and spherical states exceeds the difference of pair energies. Upon their filling, the sum $\Delta E + \Delta W$ changes sign and the nucleus acquires a deformed equilibrium form. Since the change of sign of the sum comes about for sufficiently large $\Delta E(A)$, then the equilibrium deformation of the nucleus in this case changes abruptly from zero to a value of 0.2 — 0.3.

3. DETERMINATION OF THE POSSIBLE STATES OF PROLATENESS

Up to the present time, it has not been possible to make any assumptions as to the dependence of the nuclear potential on the distance, and our conclusions do not depend on the form of $V(r)$. Now let us

TABLE II

Nucleus	β	Proton configuration	Neutron configuration	$E(A)$, $\hbar\omega_0^0$	ΔE, Mev	ΔW	$\Delta E+\Delta W$
$_{60}Nd^{146}_{86}$	0.3	$g^{10}_{9/2}$; $g^4_{7/2}\left(\frac{1}{2}+,\frac{3}{2}+\right)$; $d^2_{5/2}\left(\frac{1}{2}+\right)$; $h^4_{11/2}\left(\frac{1}{2}-,\frac{3}{2}-\right)$;	$h^{10}_{11/2}\left(\frac{1}{2}--\frac{9}{2}-\right)$; $f^4_{7/2}\left(\frac{1}{2}-,\frac{3}{2}-\right)$; $h^2_{9/2}\left(\frac{1}{2}-\right)$;	699.311	−12.364	12.6	+0.236
	0	$g^{10}_{9/2}$; $g^8_{7/2}$; $d^2_{5/2}$;	$h^{12}_{11/2}$; $f^4_{7/2}$	700.899			
$_{60}Nd^{148}_{88}$	0.3		$h^{10}_{11/2}$; $f^4_{7/2}\left(\frac{1}{2}-,\frac{3}{2}-\right)$; $h^2_{9/2}\left(\frac{1}{2}-\right)$; $i^2_{13/2}\left(\frac{1}{2}+\right)$;	710.758	−17.835	13.3	−4.535
	0		$h^{12}_{11/2}$; $f^6_{7/2}$;	713.059			
$_{60}Nd^{150}_{90}$	0.3		$h^{10}_{11/2}$; $f^4_{7/2}\left(\frac{1}{2}-,\frac{3}{2}-\right)$; $h^2_{9/2}\left(\frac{1}{2}-\right)$; $i^4_{13/2}\left(\frac{1}{2}+,\frac{3}{2}+\right)$;	722.387	−21.854	14	−7.854
	0		$h^{12}_{11/2}$; $f^8_{7/2}$;	725.219			
$_{62}Sm^{148}_{86}$	0.3	$g^{10}_{9/2}$; $g^4_{7/2}\left(\frac{1}{2}+,\frac{3}{2}+\right)$; $d^2_{5/2}\left(\frac{1}{2}+\right)$; $h^6_{11/2}\left(\frac{1}{2}-,\frac{3}{2}-,\frac{5}{2}-\right)$;		713.846	−13.084	12.6	−0.484
	0	$g^{10}_{9/2}$; $g^8_{7/2}$; $d^4_{5/2}$;		715.534			
$_{62}Sm^{150}_{88}$	0.3			725.294	−18.513	13.3	−5.213
	0			727.694			
$_{62}Sm^{152}_{90}$	0.3			736.924	−22.511	14	−8.511
	0			739.854			

Table II continued

Nucleus	β	Proton configuration	Neutron configuration	$E(A)$, $\hbar\omega_0^0$	ΔE, Mev	ΔW	$\Delta E+\Delta W$
$_{76}Os^{186}_{110}$	0.2	$g^{10}_{9/2}$; $g^8_{7/2}$; $d^6_{5/2}$; $h^{10}_{11/2}\left(\frac{1}{2}--\frac{9}{2}\right)$; $d^2_{3/2}\left(\frac{1}{2}+\right)$;	$h^{12}_{11/2}$; $f^8_{7/2}$; $h^8_{9/2}\left(\frac{1}{2}--\frac{7}{2}-\right)$; $i^{10}_{13/2}\left(\frac{1}{2}+-\frac{9}{2}+\right)$; $p^2_{3/2}\left(\frac{1}{2}-\right)$;	970.857	−10.523	7.7	−2.823
	0	$g^{10}_{9/2}$; $g^8_{7/2}$; $d^6_{5/2}$; $h^{12}_{11/2}$;	$h^{12}_{11/2}$; $f^8_{7/2}$; $h^{10}_{9/2}$; $i^{10}_{13/2}$;	972.322			
$_{76}Os^{188}_{112}$	0.2		$h^{12}_{11/2}$; $f^8_{7/2}$; $h^8_{9/2}\left(\frac{1}{2}--\frac{7}{2}-\right)$; $i^{10}_{13/2}\left(\frac{1}{2}+-\frac{9}{2}+\right)$; $p^2_{3/2}\left(\frac{1}{2}-\right)$; $f^2_{5/2}\left(\frac{1}{2}-\right)$;	983.748	−7.758	7	−0.758
	0		$h^{12}_{11/2}$; $f^8_{7/2}$; $h^{10}_{9/2}$; $i^{12}_{13/2}$;	984.832			
$_{76}Os^{190}_{114}$	0.2		$h^{12}_{11/2}$; $f^8_{7/2}$; $h^8_{9/2}\left(\frac{1}{2}-\frac{7}{2}-\right)$; $i^{10}_{13/2}\left(\frac{1}{2}+-\frac{9}{2}+\right)$; $p^4_{3/2}$; $f^2_{5/2}\left(\frac{1}{2}-\right)$;	996.673	−4.772	6.3	+1.528
	0		$h^{12}_{11/2}$; $f^8_{7/2}$; $h^{10}_{9/2}$; $i^{14}_{13/2}$;	997.342			
$_{76}Os^{192}_{116}$	0.1		$h^{12}_{11/2}$; $f^8_{7/2}$; $h^{10}_{9/2}$; $i^{10}_{13/2}\left(\frac{1}{2}+-\frac{9}{2}+\right)$; $p^4_{3/2}$; $f^2_{5/2}\left(\frac{1}{2}-\right)$;	1009.619	−3.788	5.6	+1.812
	0		$h^{12}_{11/2}$; $f^8_{7/2}$; $h^{10}_{9/2}$; $i^{14}_{13/2}$; $p^2_{3/2}$;	1010.152			

undertake the determination of the regions of deformation, making use of the entire scheme of bound states which are taken into account by starting from a model of an "oscillator potential well".[2]

Change in sign of the sum $\Delta E + \Delta W$ can be brought about by filling the lowest levels of the N-th shell. Actually, for $\beta = 0$, these levels have the highest j, equal to $N + \frac{1}{2}$, thanks to spin-orbit coupling, and usually enter into the constitution of the (N − 1)-st shell (for example, the level $i_{13/2}$ which enters into the constitution of the fifth oscillator shell). The state with $\Omega = \frac{1}{2}$, which is produced in the splitting of the term $j_{max} = N + \frac{1}{2}$, has an orbit which is elongated in comparison with the states of the (N − 1)-st shell, and its filling can lead to the formation of an interval of elongated nuclei In similar fashion, the filling of states with $\Omega = j_{max}$ can lead to the formation of an interval of oblate nuclei.

For determination of the sign of the deformations, we compare the energy of the lowest levels for $\beta > 0$ and $\beta < 0$. The energy of the level for large deformations in the model of the oscillator potential well is equal to

$$E_\Omega = \hbar\omega_0^0 (1 + {}^1/_9\varepsilon^2) [N + {}^3/_2 + {}^1/_3\varepsilon (N - 3n_z)] - C_{N\Omega n_z}, \quad (13)$$

where n_z is the oscillator quantum number along the z axis, $C_{N\Omega n_z}$ is a small constant of the state (on the order of magnitude of $0.1\,\hbar\omega_0^0$), which is independent of the deformation. The lowest states for $\epsilon > 0$ have $\Omega = \frac{1}{2}$, $n_z = N$ and $\Omega = {}^3/_2$, $n_z = N - 1$. Their energies are respectively equal to

$$E_{1/2} = \hbar\omega_0 (\varepsilon) [N + {}^3/_2 - {}^2/_3\varepsilon N] - C_{1/2}, \quad E_{3/2} = \hbar\omega_0 (\varepsilon) [N + {}^3/_2 - {}^1/_3\varepsilon (2N - 3)] - C_{3/2}. \quad (14)$$

The lowest states for $\epsilon < 0$ have

$$\Omega = N + {}^1/_2, \ n_z = 0 \ \text{and} \ \Omega = N - {}^1/_2, \ n_z = 1.$$

Their energies are respectively equal to

$$E_{N+1/2} = \hbar\omega_0 (\varepsilon) [N + {}^3/_2 - {}^1/_3 |\varepsilon| N] - C_{N+1/2}, \qquad E_{N-1/2} = \hbar\omega_0 (\varepsilon) [N + {}^3/_2 - {}^1/_3 |\varepsilon| (N - 3)] - C_{N-1/2}. \quad (15)$$

As comparison of (14) with (15) shows, for large $|\epsilon|$,

$$E_{1/2} < E_{N+1/2}, \quad E_{3/2} < E_{N-1/2}.$$

In other words, for large deformations, an elongated equilibrium shape is more advantageous and, since the transition is accomplished precisely for large deformations, the absence of nuclei with negative quadrupole moments can be explained on these grounds.

The finding of the possible intervals of elongation has been carried out by comparison of the total energies of the nuclei for $\beta = 0$ and $\beta > 0$.

(a) Region of rare earths (even-even isotopes $_{60}Nd^{146} - _{76}Os^{192}$, Table II). For each nucleus, the nucleonic configurations of the ground states were found for $\beta = 0$ and $\beta > 0$, and the sums of the single particle energies were computed in terms of the oscillator quantum $\hbar\omega_0^0 \approx 41A^{-1/3}$ Mev. The difference in the one-particle energies ΔE is equal to the difference between the pair energies in the spherical and deformed states. The latter is proportional to the number of pairs of monoenergetic nucleons, if the shell is less than half filled, and correspondingly, to the number of hole pairs if the shell is more than half filled, since, for filling the shell, the pair energies in the spherical and deformed states are identical, while the energy states of a given number of nucleons outside a filled shell are analogous to the states of the same number of holes.

The sum $\Delta E + \Delta W$ changes sign in the transition form N = 86 to N = 88, which corresponds to the filling of the neutron state $i_{13/2}(\frac{1}{2}+)$. The value obtained for the lower limit of prolateness agrees with the experimental values for the filling of the neutron states $i_{13/2}(\frac{1}{2}+, {}^3/_2+)$. In the filling of each of these states, ΔE changes by a quantity far exceeding the difference in the pair energies, which produces a transition to the prolate equilibrium form.

In the region of the lower boundary of the lower limit of prolateness, the proton subshell Z = 64 is close to being filled, as a consequence of which the protons do not affect the establishment of the prolate equilibrium form.

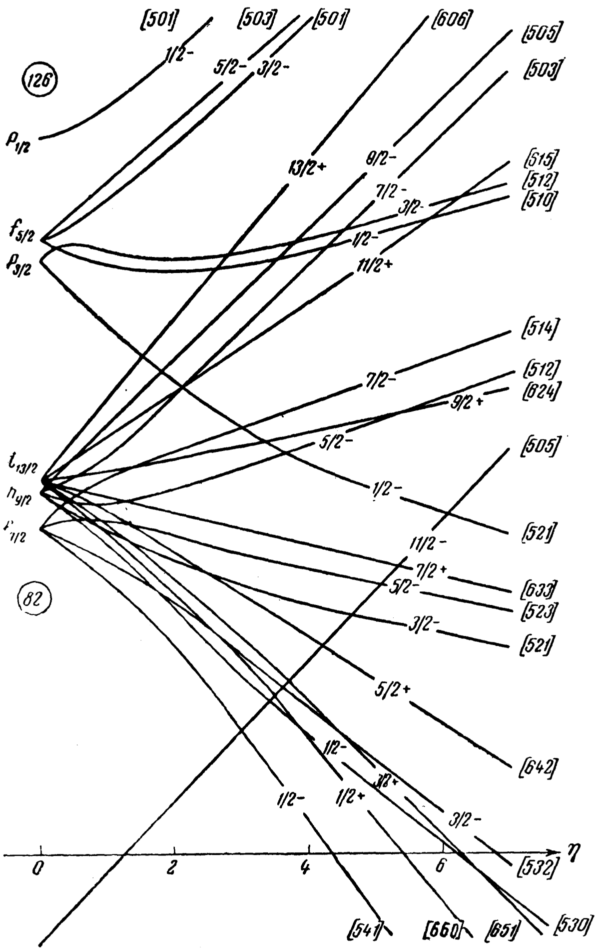
The inverse transition to the spherical equilibrium form is achieved for Z = 76, N = 116, which corresponds to a filling of the proton state $h_{11/2}({}^9/_2-)$ and the neutron states $p_{3/2}({}^3/_2-)$ and $h_{9/2}({}^9/_2-)$. The value found for the upper limit of prolatenes agrees with the experimental value Z = 76, N = 114. In contrast to the lower limit, the transition here is determined by the filling of both the neutron and the proton states, inasmuch as now both shells are close being filled.

(b) Region of heavy nuclei (Table III). The rotational region found in the heavy elements begin with the nucleus $_{88}Ra^{222}$. Inasmuch as N = 134 corresponds to filling of five oscillator shells, the neutrons do not exhibit any noticeable effect on the transition to the prolate equilibrium shape. Therefore, energy comparison is carried out for isotones with N = 134, since the problem consists of searching for proton states which govern the transition.

TABLE III

Nucleus	β	Proton configuration	Neutron configuration	$E(A)\ \hbar\omega_0^0$	ΔE, ΔW, Mev	ΔE+ΔW
$_{84}Po^{218}_{134}$	0.3	$h^{10}_{11/2};\ f^{2}_{7/2}\left(\frac{1}{2}-\right);\ i^{2}_{13/2}\left(\frac{1}{2}+\right);$	$i^{12}_{13/2};\ g^{6}_{9/2}\left(\frac{1}{2}+,\frac{3}{2}+,\frac{5}{2}+\right);\ i^{4}_{11/2}\left(\frac{1}{2}+,\frac{3}{2}+\right)$	1196.094	—4.980	+4.020
	0	$h^{12}_{11/2};\ f^{2}_{7/2};$	$i^{14}_{13/2};\ g^{8}_{9/2}$	1196.825	9	
$_{86}Rn^{220}_{134}$	0.3	$h^{10}_{11/2};\ f^{2}_{7/2}\left(\frac{1}{2}-\right);\ h^{2}_{9/2}\left(\frac{1}{2}-\right);\ i^{2}_{13/2}\left(\frac{1}{2}+\right);$		1213.152	—10.954	—1.454
	0	$h^{12}_{11/2};\ f^{4}_{7/2};$		1214.765	9.5	
$_{88}Ra^{222}_{134}$	0.3	$h^{10}_{11/2};\ f^{2}_{7/2}\left(\frac{1}{2}-\right);\ h^{2}_{9/2}\left(\frac{1}{2}-\right);\ i^{4}_{13/2}\left(\frac{1}{2}+,\frac{3}{2}\right);$		1230.327	—17.063	—7.063
	0	$h^{12}_{11/2};\ f^{6}_{7/2};$		1232.847	10	

With this aim, we constructed the Nilsson scheme for states of the proton shell 82 — 126 (see drawing). The construction was performed by the same method which was used in Ref. 2. For the state of the fifth oscillator shell, we assumed the same constant of spin-orbit coupling κ which was used in Ref. 2 for the state of the fourth proton shell (see also Ref. 4). For the states of the sixth oscillator shell, the constant of spin-orbit coupling was increased several fold in order to obtain better agreement with experiment (the values $\kappa_5 = 0.0613$, $\kappa_6 = 0.058$ were taken in place of the value $\kappa_6 = 0.05$ of Nilsson). In Table IV, the spins and parities of the ground state of nuclei with an odd number of protons are compared with the experimental values of these quantities. Agreement with experiment can be regarded as satisfactory.



The calculation carried out with the use of the drawing shows that the transition to the prolate equilibrium form is determined by filling of the proton states $i_{13/2}$ ($\frac{1}{2}$ +, $^{3}/_{2}$+), i.e., the same states which produce the transition in the region of the rare earths. The limit of the region of prolateness corresponds to Z = 88 and agrees with experiment.

TABLE IV

Chemical Symbol	Z	Theoretical value of the spin	Experimental value
Ac	89	$^3/_2$—, $^3/_2$+	$^3/_2$—
Pa	91	$^3/_2$—, $^{11}/_2$—, $^5/_2$+	$^3/_2$—
Np	93	$^5/_2$+, $^3/_2$—	$^5/_2$+
Am	95	$^3/_2$—, $^5/_2$—	$^5/_2$—
En	99	$^7/_2$+, $^{11}/_2$—	($^7/_2$)

(c) Nuclei in the region of Mg^{24}. According to the given calculation, for this region, the transition to the prolate equilibrium form is achieved by filling the state $d_{5/2}$ ($\frac{1}{2}$+) which corresponds to N, Z = 10. The reverse transition takes place upon filling of the state $d_{5/2}$ ($^5/_2$+), which corresponds to N, Z = 14. The value thus found for the lower limit of prolateness does not agree with experiment, since the rotational region begins with the nucleus $_{11}Na^{23}$. The upper limit is determined more precisely, apparently. Thus, for this region of the nuclei, the agreement of the theory with experiment is worse than in the first two; however, even here the observed rotational nuclei do not exceed the limits computed theoretically.

(d) The existence of all three intervals of elongation is brought about, as we have seen, by filling of the the lowest states of the oscillator shells. In the regions of the rare earths and heavy nuclei, the lowest states filled are ths sixth, while in the region of Mg^{24}, it is the second oscillator shell. We can expect that in the filling of the lowest states, the third, fourth and fifth shells will also produce a region of elongated nuclei. However, calculation carried out for these regions shows that the difference between the single particle energies in the deformed and spherical states does not exceed the difference of the energies of pair interaction (for the difference of pair energies we have assumed the value 1 Mev, which is the lower limit of this quantity in the regions lying about Mg^{24} and the rare earths), and consequently there are no prolate nuclei in these states.

Thus, in addition to the regions of the rare earths and of heavy nuclei, and to the region about Mg^{24}, the theory does not give other regions of prolateness, a result that agrees with the available experimental data.

CONCLUSIONS

(1) As we can see from the results of the investigation just carried out, the role of the remaining interaction of nucleons at a distance is not confined to an account of the spin dependence of nuclear forces. It is shown that the jump change of the equilibrium form of atomic nuclei, observed in experiment, is determined by the effect of the pair energy, which favors the transition to the deformed equilibrium form if number of nucleons outside the shell is small.

(2) If the difference between the pair energies in the deformed and spherical states is estimated from experimental data, than all three known regions of prolate nuclei are obtained, while the theoretical boundaries for the prolate nuclei are in agreement with the experimental data. The theory gives no evidence for the existence of other regions of prolate nuclei that do not arise from the present data.

(3) The use of the oscillator potential for the calculation of the nucleonic energies and the potential for the pair interaction does not restrict the generality of the results, inasmuch as the Nilsson scheme depends only slightly on the choice of the potential, while experimental values are used for the pair energy.

In conclusion, I consider it my pleasant duty to thank Professor L. A. Sliv for his unflagging interest in the work and for a number of valuable suggestions given during its undertaking. The author also thanks his coworkers in the Department of Theoretical Physics of the A. I. Herzen Leningrad State Pedagogical Institute for discussion of the results of the research.

APPENDIX

In Sec. 2 we determined the limiting values of the pair energies in spherical and deformed nuclear states. In the comparison with the energy of pair interaction in these states, the dependence of the pair energy on the deformation was not considered, although this effect can influence the results. Let us show that, in the case of limitingly short-range forces, the pair energy changes within a small range of variation of β between the limiting values, and consequently the approximation used is justified.

For this purpose, let us consider a system of two particles that interact according to Eq. (8). The Hamiltonian of the nucleon is expressed by Eq. (4), and for a system of two nucleons,

$$\hat{H}(1,2) = \hat{H}_N(1) + \hat{H}_N(2) + \hat{W}(1,2). \tag{1}$$

We limit ourselves to the case in which the individual moment of the nucleons j is a quantum number (the so-called j-approximation), and consider the system of two monoenergetic nucleons. In this approximation, the operator of spin-orbit coupling can be inserted, and the problem reduces to finding the eigenvalues of the equation

$$\hat{U}\Psi = u\Psi, \tag{2}$$

where

$$\hat{U} = \hat{V} + \hat{W}, \quad \hat{V} = -\beta\left[r_1 \frac{dV}{dr_1} Y_{20}(\theta_1) + r_2 \frac{dV}{dr_2} Y_{20}(\theta_2)\right], \quad W = -g\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{3}$$

We seek a solution of Eq. (2) in the form of a linear combination of eigenfunctions of the spherical part of the Hamiltonian (1)

$$\Psi = \sum_{\Omega = 1/2}^{j} B_\Omega \Psi_\Omega, \tag{4}$$

$$\Psi_\Omega = R_{Nl}(r_1) R_{Nl}(r_2) \frac{1}{\sqrt{2}} [Y_{j,\Omega}(1) Y_{j,-\Omega}(2) - Y_{j,-\Omega}(1) Y_{j,\Omega}(2)], \tag{5}$$

$Y_{j,\Omega}$ are the well known spherical harmonics in the spin. The matrix elements $\hat{V}$ and $\hat{W}$ are equal to

$$\langle \Omega_1 | \hat{V} | \Omega_2 \rangle = \sqrt{\frac{5}{\pi}}\, \beta \langle Nl | r\frac{dV}{dr} | Nl \rangle \frac{3(\Omega - 1/2)(\Omega + 1/2) - (j - 1/2)(j + 3/2)}{2j(2j+2)} \delta_{\Omega_1\Omega_2}; \tag{6}$$

$$\langle \Omega_1 | \hat{W} | \Omega_2 \rangle = \sum_{I=0}^{2j-1} D^I_{\Omega_1} D^I_{\Omega_2} w(j^2 I), \tag{7}$$

where $D^I_\Omega = \sqrt{2} C^{I0}_{\Omega;-\Omega}$; $C^{I0}_{\Omega;-\Omega}$ are the Clebesh-Gordan coefficients, and $w(j^2 I)$ is expressed by Eq. (9) of Sec. 2.

For determination of the eigenvalues of Eq. (2), we must solve a secular equation of degree $(2j + 1)/2$. We do this in the case $j = 3/2$, making use of the model of the oscillator potential well, and for definiteness setting $N = 4$, $\ell = 2$, although N and ℓ do not play an essential role in the choice of $V(r)$. The eigenvalues of (2) are then determined from the expressions

$$\begin{vmatrix} -1.4\beta - 0.18a - u & 0.12a \\ 0.12a & 1.4\beta - 0.18a - u \end{vmatrix} = 0, \tag{8}$$

where $a = g\alpha^3/4\pi\hbar\omega$ is the ratio of the constant in the energy of pair interaction to the energy of the oscillator quantum, $\alpha = \sqrt{m\omega/\hbar}$. The energy of the ground state is equal to

$$u = -0.18a - \sqrt{0.0144a^2 + 1.96\beta^2}. \tag{9}$$

in units of $\hbar\omega$. The corresponding wave function is

$$\Psi = \cos(\varphi/2)\Psi_{1/2} - \sin(\varphi/2)\Psi_{3/2}, \quad \tan\varphi = 0.12a/1.4\beta. \tag{10}$$

As is easily seen, for $\beta \to 0$

$$\Psi \to -[-\Psi_{1/2} + \Psi_{3/2}]/\sqrt{2} = -\Psi_{I=0,0}, \tag{11}$$

i.e., the total momentum $\beta = 0$ corresponds to the spherical ground state of the system for $I = 0$. For sufficiently large deformations, we can neglect the pair energy (in the limit $a \to 0$), and $\Psi \to \Psi_{1/2}$, i.e., $\Omega = \frac{1}{2}$ corresponds to the deformed ground state.

In order to obtain the pair energy as a function of the deformation, we subtract from (9) the purely deformation energy, which is obtained from (8) in the absence of the pair energy. As a result, we have

$$w(\beta) = -0.18a - 1.4[\sqrt{\beta^2 + (0.12/1.4)^2 a^2} - |\beta|]. \tag{12}$$

In the spherical state, $w = -0.3a$; in the deformed state, $w = -0.18a$. As is seen from (12), the pair energy is practically equal to the deformation value at $\beta \approx 0.1$, and for further increase of β, it is independent of the deformation.

[1] M. G. Mayer and J. H. Jensen, Elementary theory of nuclear shell structure, (New York, London, 1955).
[2] S. G. Nilsson, Dan Med. Fys. Medd. **29**, No. 16 (1955).
[3] K. Gottfried, Phys. Rev. **103**, 1017 (1956).
[4] B. Mottelson and S. Nilsson, (Probl. of Mod. Phys.) 1, 186 (1956) [presumably a translation of Z. Physik **141**, 217 (1955)].
[5] P. F. A. Klinkenberd, Revs. Mod. Phys. **24**, 63 (1952).
[6] N. Zeldes, Nucl. Phys. **2**, 1-64 (1956).
[7] J. R. Huizenga, Physica **21**, 410 (1955).
[8] A. H. Wapstra, Physica **21** 367 (1956).

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251