SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 4 APRIL, 1958

THE INFLUENCE OF A He^3 IMPURITY ON THE VISCOSITY OF HELIUM II

V. N. ZHARKOV

Institute of Terrestrial Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 11, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 929-932 (October, 1957)

On the basis of the Landau theory of liquid helium, a determination is made of the coefficient of viscosity of a weak solution of He^3 in helium II as a function of temperature and concentration.

ACCORDING to the theory of Landau, a weak solution of He^3 in helium II can be regarded as a mixture of three gases of elementary excitations:[1] (1) a gas of rotons $\epsilon_r = \Delta_r + (p - p_0)^2/2m$, (2) a gas of phonons $\epsilon = sp$, and (3) a gas of impurity excitations $\epsilon_i = \Delta_i + p^2/2\mu$.

In order to solve the problem of the viscosity of the solution it is necessary to know how the elementary excitations interact with each other. The effective cross-sections for roton-roton, roton-phonon, and phonon-phonon scattering have been calculated by Landau and Khalatnikov[2] in their well known work on the theory of the viscosity of helium II. They have also discussed the question of the establishment of energy equilibrium in a phonon gas, which plays an important role in the study of kinetic effects in helium II. Effective cross-sections for the scattering of impurities by rotons and by each other, and for the scattering of phonons by impurities, have been calculated by Khalatnikov and the author[3] for an investigation of diffusion and of thermal conduction in solutions.

From the conservation laws it follows that, relative to the phonons, the rotons and the impurity excitations can be regarded as heavy particles. Moreover, the absolute values of the effective cross-sections for the scattering of the heavy particles by each other are much greater than the cross-sections for the scattering of phonons by these particles. These facts lead to the result that, for concentrations of He^3 atoms greater than $\sim 10^{-6}$ and for all temperatures, deviations of the roton and impurity distribution functions from their equilibrium values are determined only by processes which involve the scattering of the "heavy" excitations by each other. Furthermore, in solving the kinetic equations which determine the phonon distribution function, the rotons and impurities can be described by equilibrium Maxwellian distribution functions, since deviations of these functions from equilibrium are much smaller than deviations of the phonon distribution function from its equilibrium value. Both these circumstances make the calculations extremely simple and analogous to those carried out earlier in determining the thermal conductivity κ of a solution.[3]

The viscosity, as well as the thermal conductivity, of a solution is the sum of three parts: (1) the "roton viscosity" η_r, which depends on the transfer of momentum by rotons, (2) the "phonon viscosity" η_{ph}, which depends on the transfer of momentum by phonons, and (3) the "impurity viscosity" η_i, which depends on the transfer of momentum by the impurities. Analysis of the thermal conductivity of a solution of He^3 in helium II has shown that κ_r and κ_{ph}, the roton and phonon parts of the coefficient of thermal conductivity, differ from their corresponding values κ_{0r} and κ_{0ph} for pure helium only in that the scattering processes determining their effective times are modified by the presence of impurities. As we have shown,[3] κ_r is obtained from κ_{0r} by replacing the time t_{rr}^{-1} in the latter by the time $t_r^{-1} = t_{rr}^{-1} + t_{ri}^{-1}$, where t_{rr} and t_{ri} are quantities differing from the average times between roton-roton and roton-impurity collisions by temperature-independent factors of order unity. These times are[2,3]

$$t_{rr} = \hbar^4 / 4p_0 m \mid V_0 |^2 N_r = 0.7 \cdot 10^{-13} T^{-1/2} e^{\Delta_r/kT}, \tag{1}$$

$$t_{ri}^{-1} = \overline{v_i \sigma_{ir} N_i} = 3.3 \cdot 10^{12} c T^{1/2}. \tag{2}$$

Here the usual notation is used: Δ_r and p_0 are the zero-point energy and momentum of a roton, m the effective mass of a roton, N_r the number of rotons per unit volume, v_i the velocity of an impurity excitation, σ_{ir} the effective cross-section for the scattering of an impurity by a roton, N_i the number of impurity atoms per unit volume, $c = N_3 m_3/(N_3 m_3 + N_4 m_4)$ the concentration; M_3 and m_4, N_3 and N_4 the masses and numbers of atoms of He^3 and He^4 per unit volume. The representation of t_r^{-1} in the form of a sum is justified in the limiting cases $t_{rr} \ll t_{ri}$ and $t_{rr} \gg t_{ri}$, i.e., when rotons are scattered only by other rotons or only by impurities.

Because of the fact that the quantity t_{rr}^{-1}, which is proportional to the number of rotons, decreases exponentially with decreasing temperature, the transition from the case $t_{rr} \ll t_{ri}$ to the case $t_{rr} \gg t_{ri}$ (for a given concentration) will take place in a temperature interval of the order of ~ 0.1°K. Analogously, in the case of phonons, κ_{ph} will be obtained from κ_{0ph} by replacing θ_{phr}^{-1} by $\theta^{-1} = \theta_{phr}^{-1} + \theta_{phi}^{-1}$, where θ_{phr} and θ_{phi} are times characterizing the scattering of phonons by rotons and by impurities, respectively:[2,3]

$$1/\theta_{phr} = \frac{6!N_r}{4\pi s}\left[\frac{p_0 (kT/s)^2}{\hbar^2\rho}\right]^2 A, \tag{3}$$

$$1/\theta_{phi} = \frac{6!N_i}{8\pi s}\left[\frac{(3\mu kT)^{1/2} (kT/s)^2}{\hbar^2\rho}\right]^2 \delta. \tag{4}$$

Here s is the velocity of sound and ρ the density of helium. The quantities A and δ are temperature-independent constants of order unity.

The representation of θ^{-1} in the form of a sum $\theta_{phr}^{-1} + \theta_{phi}^{-1}$ is also justified only in the limiting cases $\theta_{phr} \ll \theta_{phi}$ and $\theta_{phr} \gg \theta_{phi}$, and the formula $\theta^{-1} = \theta_{phr}^{-1} + \theta_{phi}^{-1}$ represents an interpolation. Except for the replacement of t_{rr}^{-1} by t_r^{-1} and of θ_{phr}^{-1} by θ^{-1}, the form of the coefficients of thermal conductivity κ_{0r} and κ_{0ph} remains the same. It is, therefore, natural that the relation which exists between η_{0r} and η_{0ph}, on the one hand, and between κ_{0r} and κ_{0ph}, on the other, also remains the same for the corresponding coefficients for solutions of He^3 in helium II (this fact can be confirmed also by direct calculation). This relation was established by Khalatnikov[4] and has the form

$$\eta_{0r} = \varkappa_{0r} p_0^2 T / 5\Delta_r^2; \qquad \varkappa_{0r} = \Delta_r^2 t_{rr} N_r / 3mT, \tag{5}$$

$$\eta_{0ph} = \frac{T}{5s^2}\left(1 + \frac{\overline{\theta}_{phr}}{56\tau_{ph}}\right)^{-1}\left(1 - \frac{S_0 T}{\rho_{n0} s^2}\right)^{-1}\varkappa_{0ph}, \quad \varkappa_{0ph} = N_{ph} k \theta_{phr} s^2 \left(1 - \frac{S_0 T}{\rho_{n0} s^2}\right)\begin{cases} 19\,\dfrac{1 + 0.75\,\theta_{phr}/\theta_{ph}}{1 + 8\theta_{phr}/\theta_{ph}} & (\text{for } T > 0.9^\circ\text{K}) \\ 1.8 & (\text{for } T < 0.9^\circ\text{K}), \end{cases} \tag{6}$$

where τ_{ph} is a characteristic time for the phonon-phonon scattering process, which according to Landau and Khalatnikov,[2] is equal to

$$\tau_{ph}^{-1} = \frac{3\cdot 13!\,(u+2)^4}{5\cdot 2^{19}\,(2\pi)^3\,\hbar^2\rho^2 s}\left(\frac{kT}{s}\right)^9 \left(u = \frac{2\rho}{s}\frac{\partial s}{\partial \rho}\right). \tag{7}$$

Here $N_{ph} = 2.4 \times 4\pi\,(kT/2\pi\hbar s)^3$ is the number of phonons per unit volume, θ_{ph} is the characteristic time for the establishment of equilibrium with respect to the number of phonons in the phonon gas, and S_0 and ρ_{n0} are the entropy per unit volume and the normal density of pure helium. For the temperature region $T > 0.9$°K, $\theta_{phr}/56\tau_{ph} \ll 1$ and the second term in the parentheses in (6) can be dropped. The bar over the time θ_{phr} denotes that the time characterizing the scattering of phonons by rotons has been calculated with a different dependence of the deviations of the distribution function δn on the scattering angle than in the case of the thermal conductivity; namely, for calculations of the viscosity we have $\delta n \sim \cos\theta \sin\theta \cos\varphi$. In the case of a solution, τ_{ph}^{-1} can be set equal to zero for $c > 10^{-4}$. For concentrations $c < 10^{-4}$, $\theta/\theta_{ph} \ll 1$ for $T < 0.8$°K.

Taking all the above considerations into account and with the help of (5) and (6) we find the roton and phonon parts of the viscosity coefficient of a solution:

$$\eta_r = t_r p_0^2 N_r / 15m, \tag{8}$$

$$\eta_{ph} = \begin{cases} 3.8\cdot N_{ph} kT\overline{\theta}\,\dfrac{1 + 0.75\overline{\theta}/\theta_{ph}}{1 + 8\,\theta/\theta_{ph}} \begin{cases}\text{for } T > 0.6^\circ\text{K};\ c > 10^{-4} \\ \text{for } T > 0.9^\circ\text{K};\ c < 10^{-4}\end{cases} \\ 0.35 N_{ph} kT\overline{\theta}\left(1 + \dfrac{\overline{\theta}}{56\tau_{ph}}\right) \quad \text{for } T < 0.8^\circ\text{K};\ c < 10^{-4}. \end{cases} \tag{9}$$

Here account has been taken on the fact that to determine the viscosity coefficients from the thermal conductivity coefficients it is necessary to replace the unbarred effective times by those with a bar. Such a substitution has little effect on the effective times; thus, $\bar{\theta}_{\mathrm{phr}}$ and $\bar{\theta}_{\mathrm{phi}}$ differ from θ_{phr} and θ_{phi} only because of insignificant changes of the constants A in (3) and δ in (4), which remain of order of magnitude unity.

A calculation of the impurity part of the viscosity coefficient η_{i} can be carried out in the same way as in the kinetic theory of gases in the two limiting cases $t_{\mathrm{ir}} \ll t_{\mathrm{ii}}$ and $t_{\mathrm{ir}} \gg t_{\mathrm{ii}}$. Here t_{ir} is an effective time characterizing the scattering of impurities by rotons:[3]

$$t_{i\mathrm{r}}^{-1} = \sigma_{i\mathrm{r}} \sqrt{3kT/\mu}\, N_{\mathrm{r}} = 1 \cdot 10^{13} T e^{-\Delta_{\mathrm{r}}/kT}. \tag{10}$$

The effective time t_{ii} characterizing the scattering of impurities by each other is given by the equation

$$t_{ii}^{-1} = \overline{|v_{i1} - v_{i2}|\, \sigma_{ii} N} = \frac{a}{\pi} \frac{|V_{01}|^2 \mu^2}{\hbar^4} \sqrt{\frac{kT}{\pi\mu}}\, N = 2.9 \cdot 10^{11} a T^{1/2} c, \tag{11}$$

where μ is the effective mass of an impurity excitation, σ_{ii} is the effective cross-section for the scattering of impurities by impurities,[3] and V_{01} is a constant of the interaction between impurities. A constant a has been introduced in (11) since the magnitude of V_{01} is not known. A numerical value for (11) is obtained by taking $|V_{01}|^2 \sim 10^{-76}$ erg^2 cm^6. It can be assumed that the quantity a is on the order of unity. With the aid of the interpolation formula $t_{\mathrm{i}}^{-1} = t_{\mathrm{ir}}^{-1} + t_{\mathrm{ii}}^{-1}$ we determine the effective time characterizing the scattering of impurities in the solution, and then after elementary calculations obtain

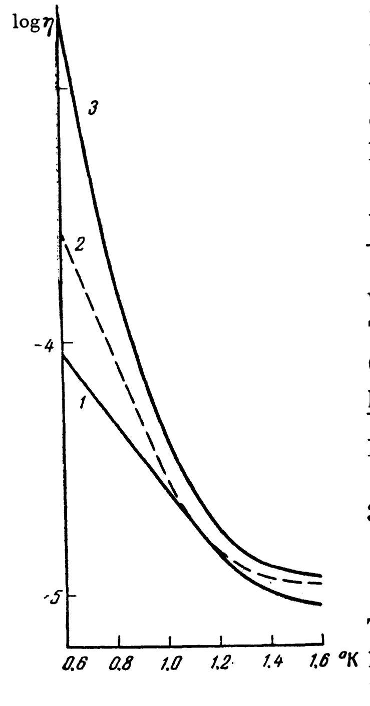
$$\eta = (15\pi/32)(kT/m_3)\, c p t_i. \tag{12}$$

After collecting the expressions obtained above, substituting numerical values for the parameters and making use of the final data given by Khalatnikov in his review article,[5] we obtain for the viscosity of the solution

$$\eta = \frac{1.15 \cdot 10^{-5}}{1 + 0.23 \cdot c e^{8.9/T}} + \frac{2.1 \cdot T^{1/2} \cdot 10^{-5}}{a + 2.9 \cdot 10 T^{1/2} c^{-1} e^{-8.9/T}} + \frac{3.75 \cdot 10^{-8} T^{-1/2} e^{8.9/T}}{1 + 3.5 \cdot 10^{-2} \delta T^{1/2} c e^{8.9/T}} \cdot \begin{cases} \dfrac{1 + 0.75\, \theta/\theta_{\mathrm{ph}}}{1 + 8\, \theta/\theta_{\mathrm{ph}}} & \left\{\begin{array}{l} \text{for } T \geqslant 0.6^\circ \mathrm{K};\ c > 10^{-4} \\ \text{for } T > 0.9^\circ \mathrm{K};\ c < 10^{-4} \end{array}\right. \\ 0.095\,(1 + \theta/56\tau_{\mathrm{ph}})^{-1}, & \text{for } T < 0.8^\circ \mathrm{K};\ c < 10^{-4}. \end{cases}$$

The temperature dependence of the viscosity for different concentrations is plotted in the figure. We see that above ~ 1.1°K the presence of impurities has slight effect in modifying the viscosity of pure helium II. Gradually, below 1.1°K, phonon-impurity scattering processes begin to play a predominant role, because of the abrupt decrease in the number of rotons, and lead to a sudden decrease in the phonon mean free path in the solution relative to its value in pure helium, and thus to a corresponding decrease in the viscosity of the solution relative to the viscosity of pure helium.

In conclusion we take advantage of the opportunity to express our gratitude to Professor I. M. Khalatnikov for discussions of this work.



Temperature dependence of the viscosity of a solution (in poise). Theoretical curves: (1) c = 10^{-2}; (2) c = 10^{-3}; (3) c = 0 (viscosity of pure helium II).

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[1] L. D. Landau and I. Ia. Pomeranchuk, Dokl. Akad. Nauk SSSR 59, 669 (1948).

[2] L. D. Landau and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) 19, 637, 709 (1949).

[3] V. N. Zharkov and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R) 32, 1108 (1957), Soviet Phys. JETP 5, 905 (1957).

[4] I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) 23, 21 (1952).

[5] I. M. Khalatnikov, Usp. Fiz. Nauk 59, 674 (1956).

Translated by W. M. Whitney
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