THE RELATION BETWEEN MATRIX ELEMENTS FOR "PARTICLES" AND "HOLES" IN NUCLEAR SHELL THEORY

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The relation between matrix elements for "particles" and "hole" configurations is considered for the case of two-particle operators which appear in a general non-central interaction. The particular cases of central, tensor, and spin-orbit forces are discussed. The relation between matrix elements of one-particle operators is briefly discussed.

A configuration of the type j^{m-n}, with $n < m/2$ (where m is the total number of states in the j shell) is called a "hole" configuration in the j-shell. The properties of states of the configurations j^n and j^{m-n} are similar in many ways. The relation between the matrix elements for the configurations ℓ^n and ℓ^{m-n}, where m is the total number of states in the ℓ-shell, was investigated by Racah[1] for atoms, for one-particle operators and for the special case of two-particle operators in which we have central pair interactions. In the present note, with this same end in view, we analyze the two-particle and one-particle interactions which occur in nuclear theory, for the cases of j–j and L–S coupling.

1. TWO-PARTICLE OPERATORS

The operator for any central or non-central interaction can be represented as [1–3]

$$V(1,2)=\{S^k(1,2)\cdot L^k(1,2)\}=\sum_{uv}\{S^k(s^{u_1}(1)\cdot s^{u_2}(2))\cdot L^k(l^{v_1}(1)\cdot l^{v_2}(2))\}, \tag{1}$$

where $\{A^k\cdot B^k\}$ is the scalar product of two tensor operators,[1] $S^k(s^{u_1}(1)\cdot s^{u_2}(2))$ is the tensor product[2,3,9] of rank k of two single-particle tensor operators of rank u_1 and u_2, acting in the spin spaces of particles 1 and 2, respectively, and $L^k(\ell^{v_1}(1)\cdot \ell^{v_2}(2))$ is a similar product of orbital operators:

$$L_q^k=\sum_p C^{v_1v_2k}_{p,q-p,q}\, l_p^{v_1}(1)\, l_{q-p}^{v_2}(2).$$

For central interactions, $k = 0$, while for non-central interactions, $k \neq 0$. For example, for tensor forces[2–5] $k = 2$, for spin-orbit forces[2,3,6,7] $k = 1$.

(a) Applying formulas of the Racah tensor algebra,[1,2,8,9] and using (1), we can always write the interaction operator in the form

$$\sum_{ij}V(ij)=\sum_{ij}\sum_r A_r\{t^r(i)\cdot t^r(j)\},$$

where r is the rank of the tensor operator in spin-orbit space. We know[1] that the expression for central pair interactions in L–S coupling has this same form. Therefore, there is a simple relation between the matrix elements for any central or non-central forces in j-j coupling for the configurations j^n and j^{m-n}, which is of exactly the same type as the relation between the matrix elements for central interactions for the configurations ℓ^n and ℓ^{m-n} (Cf. Refs. 1, 10, 11, and below). In particular, if j-j coupling is valid and there are only pair interactions, then we should get the same level schemes for Li^6 and N^{14}, Ca^{43} and Ca^{45} etc. (The last example is discussed briefly[12] in connection with the analysis of the structure of nuclei of the 1 $f_{7/2}$ shell.)

(b) The case of L–S coupling is much more complicated. We rewrite expression (1) in the form

$$\sum_{ij} V(ij) = \sum_{ij}\sum_{uv} \{S^k(s^{u_1}(i), s^{u_2}(j)) \cdot L^k(l^{v_1}(i) \cdot l^{v_2}(j))\} = \sum_{ij}\sum_{uv} V_0^{kk}(t^{u_1v_1}(i) \cdot t^{u_2v_2}(j)) = \frac{1}{2}\sum_{uv} V_0^{kk}(T^{u_1v_1} \cdot T^{u_2v_2})$$
$$-\frac{1}{2}\sum_{uv}\sum_{i} V_0^{kk}(t^{u_1v_1}(i) \cdot t^{u_2v_2}(i)). \tag{2}$$

for the case of n particles.

Here t^{uv} is a double tensor of rank u in spin space and rank v in orbital space, $V_0^{kk}(t^{u_1v_1} \cdot t^{u_2v_2})$ is the scalar part of the double tensor product $V^{kk}(t^{u_1v_1} \cdot t^{u_2v_2})$ of rank k in spin space and rank k in orbital space; $T^{uv} = \sum_i t^{uv}(i)$; the terminology is the same as in Racah's paper.[1] The purpose of the transformation (2) is to express the two-particle operator in terms of one-particle operators so that we can later make use of the relations between matrix elements of one-particle operators for the ℓ^n and ℓ^{m-n} configurations which were derived by Racah[1] and which are also discussed below. Using tensor algebra and fractional parentage coefficients[3,13,14] for calculating the matrix elements, we arrive after very involved calculations at the result:

$$\langle l^n\alpha LSJM | 2\sum_{ij} V(i,j) | l^n\alpha' L'S'JM\rangle = (-1)^{L+S'-J} W(LSL'S'; Jk)[k]\sum_{uv}(-1)^{u_1+v_1+u_2+v_2}\sum_{\alpha'' L'' S''} W(v_1v_2LL'; kL'')$$
$$W(u_1u_2SS'; kS'')\langle l^n\alpha LS \| T^{u_1v_1} \| l^n\alpha'' L''S''\rangle\langle l^n\alpha'' L''S'' \| T^{u_2v_2} \| l^n\alpha' L'S'\rangle - \langle l^n\alpha LSJM \Big| \sum_i X_0^{kk}(i) \Big| l^n\alpha' L''S''JM\rangle \tag{3}$$
$$\equiv \sum_{uv} A(u,v) - \sum_{uv} B(u,v).$$

X^{kk} is defined by the matrix

$$\langle l^1/_2 \| X^{kk} \| l^1/_2\rangle = [k]\sum_{uv}(-1)^{u_1+v_1+u_2+v_2} W(v_1v_2ll; kl)\, W(u_1u_2\,{}^1/_2\,{}^1/_2; k\,{}^1/_2)\langle l\,{}^1/_2 \| t^{u_1v_1} \| l\,{}^1/_2\rangle\langle l\,{}^1/_2 \| t^{u_2v_2} \| l\,{}^1/_2\rangle,$$

$[k] = 2k + 1$, and the other symbols are the same as Racah's.[1] To proceed further, we fix u_1, v_1, u_2, and v_2 on the right hand side of (3). When we go over from the ℓ^n configuration to ℓ^{m-n}, we see that

$$A(u,v) \to (-1)^{u_1+v_1+u_2+v_2} A(u,v), \quad B(u,v) \to -B(u,v).$$

We now present some specific cases. $k = 2$ for tensor forces, and the last term on the right of (3), i.e., $B(u, v)$, becomes zero. It is easily seen that the tensor interaction operator contains only those one-particle operators t^{uv} ($u = 1$), which change the parity of the particle state only if v is odd (for $v = 1$, it is a vector operator). Since all the matrix elements which appear in the expression $A(u, v)$ are diagonal with respect to configuration, v_1 and v_2 are even. Thus the following assertion holds for tensor forces: the matrix elements

$$\langle l^n\alpha LSJM \Big| \sum_{ji} V(ij) \Big| l^n\alpha' L'S'JM\rangle \text{ и } \langle l^{m-n}\alpha LSJM \Big| \sum_{ij} V(ij) \Big| l^{m-n}\alpha' L'S'JM\rangle$$

are the same (α may be, for example, the seniority quantum number, or the isotopic spin T, or both; the matrix elements are diagonal in T). For central forces the expressions $A(u, v)$ also coincide for the ℓ^n and ℓ^{m-n} configurations; the expressions $B(u, v)$ are easily seen to be independent of L, S, L', S': $\sum_{u,v} B(u,v) = nB_0$ for the ℓ^n configuration, $\sum_{uv} B(u,v) = (m-n)B_0$ for the ℓ^{m-n} configuration.

So for central forces the "level schemes" are the same for the ℓ^n and ℓ^{m-n} configurations, though the absolute values of the "binding energies" differ by $(m - 2n)B$. As mentioned earlier, this fact is well known.[1,10,11]

Our last example is the spin-orbit force. It is not difficult to show that for any interactions with $k = 1$, two types of operators t^{uv} occur in the expansion (2): (a) for odd v the parity of the particle state changes (for $v = 1$, the operator is a vector), (b) for odd v the parity of the state is unchanged (for $v = 1$, the operator is a pseudovector).

Thus even though the transformation properties for particular pairs $t^{u_1v_1}$, $t^{u_2v_2}$ are simple when we go from ℓ^n to ℓ^{m-n}, there is no overall simple relation. Both the tensor and spin-orbit forces give rise to spin-orbit splitting, but they behave completely differently; even if the spin-orbit forces are, roughly speaking, equivalent to a spin-orbit coupling of the type $\xi\sum_i \boldsymbol{\ell}_i \cdot \mathbf{S}_i$, with a constant ξ which increases with filling of the shell,[6,15] the tensor forces have no such clear connection with the Mayer-Jensen shell theory. All the theoretically admissible types of interaction have been enumerated in Refs. 16 and 17.

The relation between matrix elements for "particles" and "holes" can also be investigated by the same general methods for more complicated cases such as configuration mixing. The simplest example is the matrix element

$$\langle j^{m-1}\, T_1 J_1 \alpha_1,\, j_1,\, TJ\alpha \,\Big|\, \sum_{ij} V(ij) \,\Big|\, j^{m-1} T_1 J_1 \alpha_1, j_2,\, TJ\alpha' \rangle,$$

where $J_1 = j$, $T_1 = 1/2$, and α is the set of quantum numbers which are needed in addition to J and T for unique characterization of a state. This matrix element has previously been calculated[17,18] by other methods. The basis for the calculation is a formula which is related to formula (33) of Racah's paper[13] and derived in similar fashion:

$$\begin{aligned}\langle j^{n-1} T_1 J_1 \alpha_1,\, j_1,\, TJ\alpha \,\Big|\, \sum_{ij} V(ij) \,\Big|\, j^{n-1} T_1 J_1 \alpha_1,\, j_2,\, TJ\alpha' \rangle &= \delta(j_1, j_2) \langle j^{n-1} T_1 J_1 \alpha_1 \,\Big|\, {\sum_{ij}}' V(ij) \,\Big|\, j^{n-1} J_1 T_1 \alpha_1 \rangle \\ &+ (n-1) \sum_{T_2 J_2 \alpha_2 J_3 T_3} C^{j^{n-1} T_1 J_1 \alpha_1}_{j^{n-2} T_2 J_2 \alpha_2}\, C^{j^{n-1} T_1 J_1 \alpha_1}_{j^{n-2} T_2 J_2 \alpha_2}\, U(J_2 j J j_1; j J_3)\, U(J_2 j J j_2; j J_3) \\ &\times \{U(T_2\, {}^1\!/_2\, T\, {}^1\!/_2;\, {}^1\!/_2\, T_3)\}^2 \langle j j_1 J_3 T_3 \,|\, V(n-1, n) \,|\, j j_2 J_3 T_3 \rangle, \quad U(abcd; ef) = \{[e][f]\}^{1/2}\, W(abcd; ef).\end{aligned} \tag{4}$$

If $n = m$, from formula (19) of Racah's paper[13]

$$C^{j^{m-1},\, J_1 = j,\, T_1 = 1/2}_{j^{m-2} J_2 T_2} = \left\{ \frac{2}{m-1} \frac{[J_2]\,[T_2]}{[j]\,[T_1]} \right\}^{1/2}.$$

By substituting these values in (4), we obtain the simple final formula[18] after some transformations.

2. ONE-PARTICLE OPERATORS

Racah's[1] results require only slight modification (the inclusion of isotopic spin):

$$\begin{aligned}&\langle l^{m-n} \alpha LSTM_L M_S M_T \,|\, F^{uvr}_{\beta\gamma\delta} \,|\, l^{m-n} \alpha' L'S'T' M'_L M'_S M'_T \rangle \\ &= -(-1)^{u+v+T'-T} \langle l^n \alpha LSTM_L M_S - M_T \,|\, F^{uvr}_{\beta\gamma-\delta} \,|\, l^n \alpha' L'S'T' M'_L M'_S - M'_T \rangle.\end{aligned}$$

$F^{uvr}_{\beta\gamma\delta}$ is a triple tensor, r its rank in isotopic spin space, and β, γ, and δ are the indices of the components of the tensor. For operators which are diagonal in M_T (electromagnetic transitions, spin-orbit coupling operator $\xi \sum_i \boldsymbol{l}_i \cdot \mathbf{s}_i$ etc.), $\delta = 0$. For the operators of β-decay theory, $\delta \neq 0$. The state T, M_T of the configuration l^n is related to the state T, $-M_T$ of the configuration l^{m-n}, because, for example, a proton corresponds to a "proton hole" in the filled shell, i.e., to a nucleus with an odd neutron, so the sign of M_T changes. Thus the magnetic moments of nuclei with configurations l^n and l^{m-n} are the same, while their quadrupole moments have opposite signs, as is well known.

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THE ENERGY DEPENDENCE OF A SCATTERING CROSS SECTION NEAR THE THRESHOLD OF A REACTION

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The energy dependence of the cross section for an elastic scattering process X(aa)X is examined at energies near the threshold for a reaction X(ab)Y (b being a neutron). It is shown that the scattering cross section has a singularity at the threshold. From the singularity, one can obtain the spin and parity of the nucleus Y formed in the reaction, and also simplify the phase analysis of the elastic scattering.

1. INTRODUCTION

In a recently published work[1] it was shown that the cross section for the elastic scattering of protons from Li^7, $Li^7(pp)Li^7$, has a peak at an energy corresponding to the threshold of the reaction $Li^7(pn)Be^7$. The peak is about 40 kev wide and at some angles its height is 20 — 30% of the cross section at the same angles. The existence of such a noticeable anomaly in an elastic scattering cross section makes it worth while to consider in general the behavior of the cross section for the elastic scattering X(aa)X at energies near the threshold E_{thr} of the reaction X(ab)Y. Taking advantage of the fact that the energy dependence of the reaction cross section near its threshold is known, one can use the unitarity of the scattering matrix to determine the energy dependence of the phases for scattering near threshold.

It turns out that the behavior of the elastic scattering near threshold gives information not only on the scattering itself, but also on the reaction. In particular, such experiments can be used to find the spin and parity of the particles formed in the reaction, and also greatly simplify the phase analysis of the elastic scattering near threshold. We consider the simplest case first, that all particles a, X, b, Y have spin zero.

2. SPINLESS PARTICLES

We write down the wave function at energies above threshold ($E \geq E_{thr}$). At such energies, both elastic scattering X(aa)X and the reaction X(ab)Y are possible and the wave function has the asymptotic form

$$\Phi_a\Phi_X\left[e^{i\mathbf{k}_1\mathbf{r}}+\frac{1}{r}e^{ik_1r}\sum_l\frac{2l+1}{2ik_1}(S_l-1)P_l(\cos\theta)\right]-\Phi_b\Phi_Y\frac{e^{ikr}}{r}\sum_l\frac{2l+1}{2\sqrt{k_1k}}M_lP_l(\cos\theta), \quad (2.1)$$