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CORRELATIONS IN THE DISTRIBUTION OF CASCADE PARTICLES

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A cascade is considered to consist of several types of particles moving in a generally inhomogeneous medium varying with time. The particles collide with the particles of the medium and in this process are absorbed, scattered, and produce new cascade particles. The functions, determining the distribution of the particles of each type in cascades initiated by a single particle of a given type appearing at a given time with known initial position and velocity, are assumed to be known. By means of these functions, the probability of the joint presence of a given number of cascade particles at a given instant in a given cell of the particle type-position-velocity space is found. The detecting probability is calculated for detectors having a sensitivity dependent upon the type and velocity of particles as well as upon their place and time of incidence.

LET A_j ($j = 1, 2, \ldots n$) represent n various types of particles forming a cascade initiated by a single particle of a given type A_i with a given velocity **u**, which has appeared at the instant s at a given point **q**. Let

$$V(P, Q)\,dQ \quad (P = (s, \mathbf{q}, \mathbf{u}, i),\ Q = (t, \mathbf{r}, \mathbf{v}, j),\ dQ = d\mathbf{r}d\mathbf{v}) \tag{1}$$

be the probability that at the time t one particle of this cascade, of the type A_j has the radius vector between **r** and **r** + d**r** and velocity between **v** and **v** + d**v**. This probability was found for the case of a homogeneous[1] and of a multi-layer[2] medium. In the following we shall assume that the functions V are known for any inhomogeneous and time-varying medium without referring to their actual expressions. Using the notation and results of Refs. 1 and 2 we shall solve several generalized problems concerning the correlations in particle distribution.

1. Let L be a natural number. We shall find the probability

$$V(P, Q_1, Q_2, \ldots Q_L)\,dQ_1dQ_2 \ldots dQ_L \text{ or, in short } V(P, Q_l)\,dQ_l \tag{2}$$

that there are K_ℓ particles of the type $A_{j\ell}$ and with velocity between $\mathbf{v}_\ell$ and $\mathbf{v}_\ell + d\mathbf{v}_\ell$ having at the instants t_ℓ radius vector between $\mathbf{r}_\ell$ and $\mathbf{r}_\ell + d\mathbf{r}_\ell$ respectively.

We shall consider a cascade for which probability (2) is fulfilled, i.e., in which L particles of the type $A_{j\ell}$ with velocities between $\mathbf{v}_\ell$ and $\mathbf{v}_\ell + d\mathbf{v}_\ell$ have radius vectors between $\mathbf{r}_\ell$ and $\mathbf{r}_\ell + d\mathbf{r}_\ell$ at the instants t_ℓ. A certain number of cascade development lines C_ℓ, originating in the point P and passing through or terminating in the points Q_ℓ correspond to those particles. All the lines form a tree-like system R in the space E of the variables t, **r**, **v**, j, beginning with a single branch D_1 emerging from the point P at the time s, which then splits at the time τ_1 into two or more branches D_2, D_3, ... These, in turn, branch out at the moments τ_2, τ_3... etc. until all branches terminate in the end points Q_1. We shall denote the branching-out points of the lines D_h by P_k^*. These points will be said to be of multiplicity M(M = 2, 3, ...) when M new branches start from a point.

The system R is not continuous since the types and velocities of particles change abruptly in collisions. The projection of R in the t-**r** space is, however, a continuous branching curve. We shall refer to the set of all systems R corresponding to a given position of the points P and Q_ℓ, the projections of which in the t-**r** space are topologically equivalent, as to the graph S. A graph corresponding to the points, P, $Q_1Q_2, \ldots Q_{10}$, $P_1^* \ldots P_4^*$ is shown in Fig. 1. The points P_1^*, P_3^*, and P_4^* are double, and P_2^* is quadruple. The number N_L of all possible graphs corresponding to a given L is, obviously, finite. For example, $N_1 = 1$, $N_2 = 2$, $N_3 = 3$, etc. All graphs corresponding to the cases L = 1,2,3, are shown in Fig. 2. We shall call a graph S elementary, and denote it by T, if its branches D_h do not pass beyond the points P_ℓ but terminate in them. Every graph S can be decomposed into elementary graphs T, chosing as dividing points those of the points P_ℓ traversed by S. It is obvious that the number of elementary graphs into which every S graph can be divided is also finite—always less than L. The graph represented in Fig. 1 is divided into four elementary graphs by the points Q_1, Q_2, and Q_6.

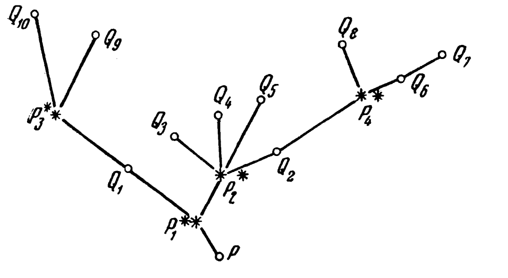

FIG. 1

We shall denote $V_S(P, Q_1)\, d\mathbf{r} d\mathbf{v}$ the probability (2) with the additional condition that cascade develop according to a given graph S. From elementary theorems on sum and product of probabilities it is evident that the function V is equal to the sum of the functions V_S corrsponding to all graphs S for a given choice of the points P and Q_ℓ, and the function V_S is equal to the product of functions corresponding to all elementatry graphs T into which S may be decomposed.

$$V = \sum_S^L \prod_T^S V_T. \tag{3}$$

Consequently, the solution is reduced to a calculation of the functions V_T corresponding to the elementary graphs.

It is easy to see from the rules for the sum and product of probabilities that V_T represents a sum of multiple integrals, the integrands of which the products of the following factors: a factor

$$V(P', P'')\, dP'' \quad (P' = (\tau', \rho', \mathbf{v}', k'),\ P'' = (\tau'', \rho'', \mathbf{v}'', k'')),$$

corresponds to every branch D with the origin in Q′ and end in Q″, and a factor

$$Q^M(P^*, P_m^*)\, d\tau d\mathbf{w}_m \quad (P^* = (\tau, \rho, \mathbf{v}, k),\ P_m^*(\tau, \rho, \mathbf{w}_m, l_m)),$$

corresponds to every branching point P* of multiplicity M. ℓ_m and $\mathbf{w}_m$ denote the types and velocities of secondary particles (m = 1, 2 ... M). This factor represents the probability that the incident particle will collide with the time $d\tau$ and that among secondary particles there will be M particles of the type ℓ_m with velocity between $\mathbf{w}_m$ and $\mathbf{w}_m + d\mathbf{w}_m$ respectively. Integration and summation is carried out over τ, ρ, ν, k characterizing all the branching points and over the types and velocities of all secondary particles.

Assuming without loss of generality that $t_1 < t_2$, we find, for instance, for L = 2:

$$V(P, Q_1, Q_2) = V(P, Q_1)\, V(Q_1, Q_2) + \sum_{P^*, l_1, l_2} \iint V(P, P^*)\, Q^2(P^*, P_1^*, P_2^*)\, V(P_1^*, Q_1)\, V(P_2^*, Q_2)\, d\mathbf{w}_1 d\mathbf{w}_2.$$

where Σ denotes integration over τ, ρ, ν, and summation over k.

2. In solving the above problem we did not impose any limitation upon either the type of the development graph or the number of collisions in each of its branches. We can modify the problem assuming that the cascade develops according to a given graph and that the number of collisions in some of its branches is pre-determined. It is easy to see that instead of relation (3) we shall have now

$$V = \prod_{T}^{S} V_T. \tag{4}$$

The factor in Eq. (4) corresponding to each branch with a fixed number of collisions will now be, instead of $V(P', P'')$,

$$\sum_{P''_g l_g} \int W(P', P''_1) \prod_g Q'(P''_g, P'_g)\, W(P'_g, P''_{g+1})\, dw_g$$

$$(P''_g = (\tau_g, \rho_g, \nu_g, k_g),\ P'_g = (\tau_g, \rho_g, w_g, l_g),\ g = 1, 2, \ldots G-1),$$

where Q' and W are given by Eqs. (5) and (6) of Ref. 1 respectively.

If we want to calculate, for example, $V_T(P, Q_1, Q_2, Q_3)$ where T is the last graph of Fig. 2, under the condition that the number of collisions along the first branch is arbitrary and along all other four branches limited to one per branch, we obtain

$$V_T(P, Q_1, Q_2, Q_3) = \sum_{P_1^* P_2^* l_{11} l_{12} l_{21} l_{22}} \iiiint V(P, P_1^*)$$
$$\times\, Q^2(P_1^*, P_{11}^*, P_{12}^*)\, W(P_{11}^*, P_2^*)\, Q^2(P_2^*, P_{21}^*, P_{22}^*)$$
$$\times\, W(P_{21}^*, Q_1)\, W(P_{22}^*, Q_2)\, W(P_{12}^*, Q_3)\, dw_{11} dw_{12} dw_{21} dw_{22}.$$

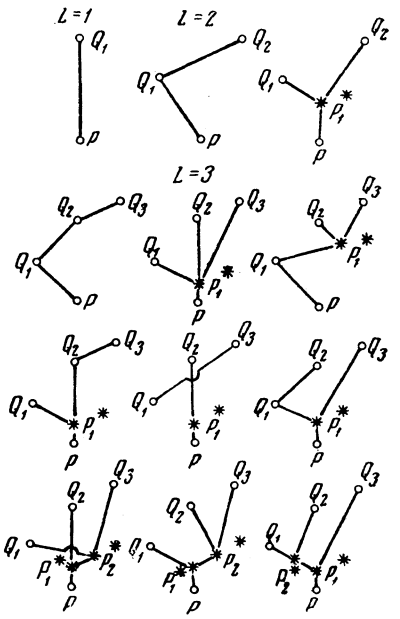

FIG. 2

3. In deducing the expressions (3) and (4) we assumed that the regions in which the particles P_ℓ are found are infinitesimally small. We shall generalize the problem for the case when the regions are finite. Let $t_\ell (\ell = 1, 2, \ldots, L)$ be L moments of time, let R_ℓ be L regions in the $\mathbf{r} - \mathbf{v} - \mathbf{j}$ space, and let N_ℓ be L positive integers. We shall denote by

$$W(P, t_l, R_l, N_l) \tag{5}$$

the required probability of finding N_ℓ particles at the moments t_ℓ and in the regions R_ℓ respectively.

Let M_ℓ be L positive integers and let $\mathbf{r}_{\ell m}, \mathbf{v}_{\ell m}, \mathbf{j}_{\ell m} (m = 1, 2, \ldots, M_\ell)$ represent the coordinates of M_ℓ points in the region R_ℓ. Let

$$V(P, Q_{lm})\, dQ_{lm} \quad (l = 1, 2, \ldots L,\ m = 1, 2, \ldots M_l)$$

be the probability of the type (2) for $L = \Sigma M_\ell$ and $t_{\ell m} = t_\ell$.

We shall denote by

$$V(P, t_l, R_l, M_l) = \frac{1}{\prod_l M_l!} \sum_{i_{lm}} \int V(P, Q_{lm})\, dQ_{lm} \tag{6}$$

the probability that there will be at least M_ℓ particles in the regions R_ℓ at the moments t_ℓ respectively. [In Eq. (6) and in the following $dQ_{\ell m} = dr_{\ell m} dv_{\ell m}$, and the integration is carried out over the part of the $\mathbf{r}_{\ell m}\mathbf{v}_{\ell m}$ plane contained in R_ℓ].

In order to find the relation between (5) and (6) we shall solve first an auxiliary problem. Let there be given I cells $C_i (i = 1, 2, \ldots, I)$ such that there can be at most one particle in each of them. Let the appearance of particles in the different cells be uncorrelated. Let P and Q be positive integers not greater than I. Let $i_p (p = 1, 2, \ldots, P)$ and $j_q (q = 1, 2, \ldots, Q)$ be two sequences of different natural numbers $\leq I$. We shall denote by $V(i_p)$ the probability that the cells C_i are occupied regardless of whether the other cells are occupied or not, and by $W(j_q)$ the probability that the cells C_{jp} are occupied while all other cells are free. Both these functions are symmetric with respect to all arguments. It is evident that

$$V_P = \frac{1}{P!} \sum_{i_p} V(i_p) \quad \text{and} \quad W_Q = \sum_{j_q} W(j_q) - \tag{7}$$

represent, respectively, the probability that not less than P or exactly Q cells are occupied. We shall show that the following relation holds:

$$V(i_p) = \frac{1}{(Q-P)!} \sum_{j_q}^{q>P} W(j_q) \quad (j_q = i_q \text{ for } q \leqslant P), \tag{8}$$

where the summation is carried out over all j_q for $q < P$. Eq. (8) expresses the elementary fact that the probability that the cells C_{ip} are occupied regardless whether other cells are occupied equals the sum of the occupation probabilities for all combinations of the cells C_i containing the cells C_{ip}. The factor $1/(Q-P)$ is due to indeterminate grouping of the numbers j_q. Summing over all i_p and taking into account Eq. (7), the relation (8) leads to

$$V_P = \sum_{Q=P}^{\infty} \frac{Q!}{Q!(Q-P)!} W_Q \quad (P = 0, 1, \ldots).$$

This equation represents an infinite system of equations for W_Q which can be solved easily, yielding

$$W_Q = \sum_{P=Q}^{\infty} \frac{(-1)^{P-Q} P!}{Q!(P-Q)!} V_P \quad (Q = 0, 1, \ldots). \tag{9}$$

The values (5) and (6) are analogous to those of Eq. (9). In consequence they are connected by a relation similar to (9)

$$W(P, t_l, R_l, N_l) = \sum_{M_l=N_l}^{\infty} \prod_{l=1}^{L} \frac{(-1)^{M_l-N_l} M_l!}{N_l!(M_l-N_l)!} V(P, t_l, R_l, M_l). \tag{10}$$

Equation (10) is more complicated than Eq. (9) since we are dealing now with L regions R_ℓ instead of one.

In calculating the probability (5) we did not require the N_ℓ particles found in R_ℓ to be of a certain type. The solution, however, is more general — it is sufficient to divide the region R into n regions $R_{\ell j}$ in order to find the probability that given numbers of particles $N_{\ell j}$ (j = 1, 2, ... n) of each type A_j are present in R_ℓ.

4. We can generalize the obtained solution further if we replace the regions R_ℓ by detecting devices, acting at the moments t_ℓ respectively, each of which detects the traversing particles with a certain probability $U^\ell(Q)$ depending on the device R_ℓ, on the time t, on the type A_j of the particle, on its velocity v, and on its position **r**. The probability (5) is in this case expressed again by Eq. (9), V being given now, however, not by Eq. (6) but by a more general formula

$$V(P, t_l, R_l, M_l) = \frac{1}{\prod_l M_l!} \sum_{i_{lm}} \int V(P, Q_{lm}) \prod_{lm}^{LM_l} U^l(Q_{lm})\, dQ_{lm}. \tag{11}$$

5. The various types of counters used for the detection of cosmic rays do not fall exactly into the category of devices of Sec. 4. The actual detectors function continuously or during a certain time interval, while above it was assumed that the particles were detected at certain instants. We shall consider therefore another type of devices $R_\ell (\ell = 1, 2, \ldots L)$ which can time the arrival of particles. Each device is characterized by a certain working volume R_ℓ, which, for the sake of symmetry, we shall consider as a volume in the space E, and by a certain function $U^\ell(Q)$ defined on a surface S^ℓ of the region R_ℓ or, which is equivalent, on surfaces S_j^ℓ in E, since $U_j^\ell d\sigma$ yields the detection probability of a particle of the type A_j entering R_ℓ through an element $d\sigma$ of the surface S_j^ℓ corresponding to the values σ and $\sigma + d\sigma$ of the six-dimensional variable σ. Evidently, this assumption represents an approximation since in view of the finite resolving power of the detectors, the detection probability U^ℓ can also depend on the sequence of arrival of various particles at R_ℓ. In this approximation the probability V is again given by Eq. (11), it is necessary, however, to write according to Eq. (4) of Ref. 2, instead of $V(P, Q_{\ell m})$, the following function:

$$V^*(P, Q_{lm}) = V(P, Q_{lm}) \begin{vmatrix} 1 & \mathbf{v}_{lm} & \mathbf{F}_j(t_l, \mathbf{r}_{lm}, \mathbf{v}_{lm}) \\ \frac{\partial t_l}{\partial \sigma} & \frac{\partial \mathbf{r}_{lm}}{\partial \sigma} & \frac{\partial \mathbf{v}_{lm}}{\partial \sigma} \end{vmatrix},$$

and the integration and summation should be carried out over the surfaces S^{ℓ}.

V being known, we can find the required probability (5) by means of relation (9) which remains valid.

All the probabilities found above may be useful, for example, in investigations of the cosmic radiation by means of coincidence counters or cloud chambers.[3]

It should be noted that the probabilities found are analogous to the probabilities of certain configurations of molecules in a gaseous medium. In the case studied above, however, the problem is greatly simplified since all the probabilities can be expressed by means of the distribution function (1), while there are no similar expressions for the correlation function in gasses.[4]

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POLARIZATION OF NUCLEONS ELASTICALLY SCATTERED AGAINST TARGET PARTICLES OF SPIN 1

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The average values of the spin operators for a system of particles having spin 1 and 1/2 are calculated. The transition matrix M is given explicitely. Consideriation is given to the case of small energies, when one can restrict oneself to S- and P-waves. Expressions are obtained for the cross-section, polarization, and correlation function. Relationships are established between the parameters of the transition matrix and the experimentally observed values. A group of experiments is suggested which could enable one to determine, through triple-scattering, the amplitude of the scattered wave and to carry out a phase-shift analysis.

THE theory of reactions involving polarized nucleons has been recently developed in a series of articles.[1] The polarization arising in nucleon-nucleon collision is due to spin-orbit interaction, and its measurement provides additional information about the coefficients of the amplitude for nucleon-nucleon scattering. A group of experiments is indicated which would allow one to determine the nucleon-nucleon scattering amplitude and to carry out a phase-shift analysis.

The present article is concerned with the elastic scattering of nucleons against a target made up of spin 1 particles.

The state of the system is described as usual through the Neuman density matrix ρ in the combined spin space of the system of two particles, or through the density matrix for two independent beams of free