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THEORY OF A DEGENERATE ELECTRON LIQUID

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Landau's theory of a Fermi liquid[1] is extended to the case of an electron liquid.

1. INTRODUCTION

THE electron theory of metals successfully uses for the description of many metallic properties the representation of the electrons as independent particles, and thus considers the electrons of the metal as a gas. True, since the electrons are located in the field of the metal lattice, their properties are considerably different from those of free electrons. This difference, in particular, is characterized by the dispersion law, i.e., by the dependence of the electron energy on their quasi-momentum. The difference in the energy of the electrons in the metal from the energy of free electrons is also due to the interaction between particles. This interaction is not at all small. Actually, the mean energy of Coulomb interaction of metal electrons is on the order of their mean kinetic energy. Therefore, it is natural to expect a substantial correlation in the motion of the electrons.

Actually, in the different attempts to calculate electron correlation (cf. Ref. 2) there was discovered a substantial effect of electron correlation on many metal properties. However, all these attempts were based on the use of a microscopic picture of the metals, and, in fact, were aimed at solving the many-body problem. The absence of a small parameter from these theories, due in the first place on the equal order of magnitude of the kinetic and potential energies of the electron, does not make these theories very valuable. Therefore, it is of considerable interest to construct a phenomenological theory sufficiently complete to take into account electron correlation, and not restricted to the narrow framework of some assumption, often far from justified, which one has to make in attempting to construct a microscopic theory of many particles. Such a phenomenological theory is the theory of a Fermi liquid, formulated by Landau for a quantum liquid such as He^3.

The object of the present work is to extend the Landau theory to include a degenerate electron liquid.

Let us note that in the kinetic theory of metal electrons the correlation is not calculated in practice. The Boltzmann kinetic equation is usually used instead. Moreover, partial calculation of electron correlations originating at small distances and leading to collisions is carried out by means of the electron-electron collision integral. Actually, these collisions play a small role,[3] and therefore make a slight contribution to the correlation. On the other hand, calculation of the correlation due to the identity of the electron shows that an appreciable change results in the kinetic equations.[4] What is important is that the influence of correlation does not reduce merely to collisions. It is therefore necessary to formulate a kinetic equation for electrons correctly taking into account the correlation of the particles. It will be shown below that the Landau theory allows one to carry out a phenomenological calculation of electron correlation. As will become evident from the following arguments, the theory of a Fermi liquid, formulated in Ref. 1 presupposes a small radius of effective force; it becomes therefore necessary to examine the variations resulting from Coulomb forces acting between electrons at an infinite radius.

In Secs. 2 and 3 it is shown that distant forces are correctly described by the self-consistent equations of the electromagnetic field.

2. LANDAU THEORY OF FERMI LIQUID AND THE HARTREE-FOCK APPROXIMATION

In order to understand how to extend the Landau theory to the case of electron interaction we first compare this theory with an approximate examination of a Fermi liquid based on the Hartree-Fock approximation. For simplicity, this comparison is made for the kinetic equations in the linear approximation, i.e., for slight deviations from the homogeneous and isotropic equilibrium state. In this case the kinetic equations, obtained in Hartree-Fock approximation for a system of half-spin particles, has the following form:[4]

$$\frac{\partial\varphi}{\partial t}+\left\{\frac{\mathbf{p}}{m}-\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\frac{\partial f_0}{\partial p'}\right\}\frac{\partial\varphi}{\partial\mathbf{r}}+\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{r}}\left\{\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\varphi(\mathbf{r},\mathbf{p})-\int d\mathbf{p}'d\mathbf{r}'\varphi(\mathbf{r}',\mathbf{p}')U(\mathbf{r}-\mathbf{r}'\mid)\right\}=0, \tag{1}$$

$$\frac{\partial\sigma}{\partial t}+\left(\left\{\frac{\mathbf{p}}{m}-\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\frac{\partial f_0}{\partial \mathbf{p}'}\right\}\frac{\partial}{\partial\mathbf{r}}\right)\sigma+\left(\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{r}}\right)\left\{\frac{1}{2}\int d\mathbf{p}'\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)\sigma(\mathbf{r},\mathbf{p}')=0. \tag{2}$$

Here f_0 is the distribution function of the equilibrium state, ϕ is the non-equilibrium addition to the distribution function, σ is the spin distribution function (it is assumed that in the equilibrium state $\sigma_0 = 0$), $U(r)$ is the interaction energy of two Fermi particles, and $\nu(k) = \int d\mathbf{r}\, U(r) \exp(i\mathbf{kr})$. The spin distribution function is defined here by the relation

$$\sigma(\mathbf{r},\mathbf{p})=\mathrm{Sp}_\sigma\hat{\sigma}\hat{n}(\mathbf{r},\mathbf{p}), \tag{3}$$

where $\hat{\sigma}$ is the spin operator and $\hat{n}$ the density matrix.

Equation (2) is of the same form as the corresponding equation of the Landau theory, but Eq. (1) is substantially different since it contains a term with the particle-interaction energy U, which makes Eq. (1) integral in coordinate space. Such a difference can be removed by assuming the radius of the force effective between particles of the Fermi liquid to be small in comparison with the characteristic dimensions of the spatial inhomogeneities. In the case of a Fermi liquid such as He^3, this requirement, imposed on the radius of action of the force, is known to be satisfied. Then

$$\int d\mathbf{p}'d\mathbf{r}'\varphi(\mathbf{r}',\mathbf{p}')U(|\mathbf{r}-\mathbf{r}'|)\approx\int d\mathbf{p}'\varphi(\mathbf{r},\mathbf{p}')\int d\mathbf{r}'U(\mathbf{r}')\equiv\nu(0)\int d\mathbf{p}'\varphi(\mathbf{r},\mathbf{p}'). \tag{4}$$

Using relation (4) we can transform Eq. (1) to the following form

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial\mathbf{r}}\left\{\frac{\mathbf{p}}{m}+\frac{1}{2}\mathrm{Sp}_\sigma\frac{\partial}{\partial\mathbf{p}}\int d\mathbf{p}'\hat{f}(\mathbf{p},\mathbf{p}')f_0(\mathbf{p}')\right\}-\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{r}}\frac{1}{2}\mathrm{Sp}_\sigma\int d\mathbf{p}'\hat{f}(\mathbf{p},\mathbf{p}')\varphi(\mathbf{r},\mathbf{p}')=0, \tag{5}$$

where

$$\hat{f}(\mathbf{p},\mathbf{p}')=\nu(0)-\nu\left(\left|\frac{\mathbf{p}-\mathbf{p}'}{\hbar}\right|\right)(1+\hat{\sigma}\hat{\sigma}')/2. \tag{6}$$

If the form of the function $\hat{f}(\mathbf{p},\mathbf{p}')$ is not specified, we can say that Eq. (5) agrees with the corresponding equation of the Landau theory. In Ref. 1 the function $\hat{f}(\mathbf{p},\mathbf{p}')$ is defined as the forward scattering amplitude of two quasi-particles of a Fermi liquid taken with a negative sign. In the Hartree-Fock approximation, according to Eq. (6), this effectively turns out to be so, but only for the scattering amplitude calculated in Born approximation.

From the above it is clear that the Landau theory of a Fermi liquid does not take into account the effects possible in the case of a force with effective radius comparable with the characteristic dimensions of an inhomogeneous liquid. Therefore, that theory cannot be directly applied to the case of Coulomb interaction, which is significant for electrons.

3. ENERGY AS A FUNCTIONAL OF THE DISTRIBUTION FUNCTION IN THE CASE OF COULOMB INTERACTION OF PARTICLES

The characteristic feature of the Landau theory of a Fermi liquid, which distinguishes it from gas theory, is that, owing to considerable self-consistent interaction of the particles the energy of an individual particles depends on the state of the surrounding particles. This, naturally, makes the energy of a system of particles unequal to the sum of the energies of the individual particles, and makes it some function of the distribution function.

Actually, in the Landau theory the variation of the energy density of a system of particles for infinitely small variation of the distribution function has the form

$$\delta E(\mathbf{r})=\mathrm{Sp}_\sigma\int\varepsilon(\mathbf{p},\mathbf{r})\delta n(\mathbf{p},\mathbf{r})d\mathbf{p}. \tag{7}$$

This formula, in fact, defines the energy (Hamiltonian function) of a quasi-particle, which differs from a free particle owing to the self-consistent interaction with surrounding particles.

Further, an essential role is played in the theory of a Fermi liquid by the variation in $\epsilon(\mathbf{p}, \mathbf{r})$ due to the variation in n. In Ref. 1 it is assumed that

$$\delta\varepsilon(\mathrm{p}, \mathrm{r}) = \mathrm{Sp}_{\sigma'}\int \hat{f}(\mathrm{p}, \mathrm{p}')\,\delta n(\mathrm{p}', \mathrm{r})\,d\mathrm{p}'. \tag{8}$$

The second variational derivative of the energy density $\hat{f}(\mathbf{p}, \mathbf{p}')$ determines many properties of the liquid, and in particular, it modifies substantially the kinetic equation.

In the preceding section it was shown that it is not possible to use the Landau theory directly in the case of a force having a large effective radius. This is connected with the use of Eq. (8) in this theory. One can understand the limited scope of this equation, for example, by use of Hartree-Fock approximation. In the general case one can say that

$$\delta\varepsilon(\mathrm{p}, \mathrm{r}) = \mathrm{Sp}_{\sigma'}\int \hat{F}(\mathrm{p}, \mathrm{p}'; \mathrm{r}, \mathrm{r}')\,\delta n(\mathrm{p}', \mathrm{r}')\,d\mathrm{p}'d\mathrm{r}'. \tag{9}$$

In other words, the second variational derivative of the energy density of a liquid with respect to the distribution function depends not only on $\mathbf{p}$ and $\mathbf{p}'$ and spins, but is also a function of the coordinates $\mathbf{r}$ and $\mathbf{r}'$.

In particular, if we use the Hartree self-consistent approximation for particles interacting in a central-force field with potential energy $U(|\mathbf{r} - \mathbf{r}'|)$, then the function $\hat{F}$ assumes the following form:

$$\hat{F}_{\mathrm{X}}(\mathrm{p}, \mathrm{p}'; \mathrm{r}, \mathrm{r}') = U(|\mathrm{r} - \mathrm{r}'|). \tag{10}$$

In the Hartree approximation we disregard correlation completely since the distribution function of the system is taken in the form of the products of the distribution functions of the individual particles. The difference $(\hat{F} - \hat{F}_{\mathrm{X}})$ is therefore entirely due to the effect of the correlation of the particles. In the general case little can be said about this quantity. Note, however, that the case of practical interest, as shown below, is the one in which the radius of correlation of the particles is much less than the distance over which the distribution function varies significantly. In this case it is possible to take

$$\hat{F}(\mathrm{p}, \mathrm{p}'; \mathrm{r}, \mathrm{r}') - \hat{F}_{\mathrm{X}} \approx \delta(\mathrm{r} - \mathrm{r}')\,\hat{f}_1(\mathrm{p}, \mathrm{p}'). \tag{11}$$

Correspondingly, Eq. (9) takes the following form

$$\delta\varepsilon(\mathrm{p}, \mathrm{r}) = \mathrm{Sp}_{\sigma'}\int U(|\mathrm{r} - \mathrm{r}'|)\,\delta n(\mathrm{p}', \mathrm{r}')\,d\mathrm{p}'d\mathrm{r}' + \mathrm{Sp}_{\sigma'}\int \hat{f}_1(\mathrm{p}, \mathrm{p}')\,\delta n(\mathrm{p}', \mathrm{r})\,d\mathrm{p}'. \tag{12}$$

If the forces have a small effective radius, then Eq. (2) almost agrees with Eq. (8) taken in the Landau theory. Since in the case of He^3 the effective radius of forces as well as the radius of correlation of particles are both of the order of the interatomic distances, it is clear from the above that the use of Eq. (8) for liquid He^3 is entirely valid.

We go now to the concrete case of electrons. It is clear that the term in Eq. (12) containing the energy $U(r)$, equal in our case to e^2/r, leads, in general, to substantial difference from Eq. (8) which is the usual one for the Landau theory. However, to use Eq. (12) it is first necessary to justify approximation (11). Therefore, we shall now examine the question of electron correlation.

The correlation of degenerate electrons is due, in particular, to the sameness of the particles. This correlation is taken into account, for example, in the Hartree-Fock approximation. It is therefore possible to say at once that the distance in which the correlation is significant is of the order of the interelectron distance. Further, for electrons, for example of a metal, an important correlation is that due to their interaction (force correlation). This correlation at large distances is similar to the Debye correlation of particles, for example in electrolytes, and at small distances it is connected with the strong repulsion between electrons.

An essential feature of this correlation, for electron densities prevailing in a real metal, is that the characteristic distance in which the correlation is significant proves to be not much more than the interatomic distance.[2] It is therefore possible to say that, depending on the anisotropy, approximation (11) and Eq. (12) can be used for metal electrons. In this last relation the effect of distant forces is taken into account in the term containing the interaction energy of a pair of electrons $U(r) = e^2/r$. Calculation of the near correlation gives the function $\hat{f}_1$. The explicit form of this function is unfortunately unknown.

Apparently, it is only possible to say that the relative contribution due to the function $\hat{f}_1$ is determined by the parameter $e^2/\hbar v_0$ where v_0 is the velocity of the electrons on the Fermi surface.

Let us note a particular feature of Eq. (12) in the case $U(r) = e^2/r$. The first term on the right side of Eq. (12), if we consider it literally, leads in this case to a divergent integral for uniform distributions. This is caused by the fact that in the analysis of the system of electrons it is necessary to introduce the background of positive charges which compensate for the charge of the electrons and which correspond to the electronic charge of the ions. Therefore, in the first term on the right side of Eq. (12) we must take δn to mean the deviation of the distribution function from its spatially-homogeneous value. Hence, it follows that for a homogeneous spatial distribution of electrons Eq. (12) actually does not differ from the corresponding equation of the Landau theory, and, in particular, our function $\hat{f}_1$ coincides with the function $\hat{f}$ of the Landau theory. This means that Landau's theory of a Fermi liquid as applied to a spatially homogeneous state is suitable also in the case of Coulomb interaction between particles. The difference due to Eq. (12) arises only in the case of spatially-inhomogeneous distribution.

4. KINETIC EQUATION FOR THE ELECTRONS OF A DEGENERATE ELECTRON LIQUID

Usually the Boltzmann kinetic equation is used for the electrons in a metal. In this case the Lorentz force, which occurs in that equation, is determined by the self-consistent electromagnetic field.* In other words, the electromagnetic field equations have the following forms:

$$\operatorname{curl}\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} = 0, \qquad \operatorname{div}\mathbf{E} = 4\pi e\,\mathrm{Sp}_\sigma\int\hat{n}\,d\mathbf{p} \qquad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = 4\pi e\,\mathrm{Sp}_\sigma\int\mathbf{v}\hat{n}\,d\mathbf{p}. \tag{13}$$

In these equations $\mathrm{Sp}_\sigma\hat{n}$ represents exactly that distribution function for which the Boltzmann equation is usually written.

Such a self-consistent approach neglecting the collision integral exactly corresponds to the Hartree approximation. In the preceding section where we examined the Coulomb interaction of electrons we arrived at the result that the Hartree approximation accurately described the long-range part of the interaction of particles. Therefore, it is clear that the longitudinal part of the Lorentz force should be determined by the self-consistent electromagnetic field. Obviously, the long-range part of the electron interaction due to the transverse field should be reckoned in an analagous manner. Consequently, the long-range part of the electron interaction is accurately taken into account by the usual self-consistent kinetic equation.

The difference from usual kinetic equation arises from the neighboring correlation of the particles described by the second term of the right side of Eq. (12). However, since this term is entirely similar to Eq. (8) usually used in the Landau theory, the difference between the kinetic energy of the electrons of a degenerate electron liquid and the Boltzmann kinetic equation is, in fact, contained in the kinetic equation of the Landau theory. Therefore, the sought-for kinetic equation can be obtained from the results of Ref. 1. Then, disregarding spin-orbit interaction, it is possible to write the following kinetic equation for the electrons of a degenerate Fermi liquid:

$$\begin{gathered}\frac{\partial\hat{n}}{\partial t} + \frac{1}{2}\left(\frac{\partial\varepsilon_1}{\partial_{\mathbf{p}}}\frac{\partial\hat{n}}{\partial\mathbf{r}} + \frac{\partial\hat{n}}{\partial\mathbf{r}}\frac{\partial\varepsilon_1}{\partial\mathbf{p}}\right) - \frac{1}{2}\left(\frac{\partial\varepsilon_1}{\partial\mathbf{r}}\frac{\partial\hat{n}}{\partial\mathbf{p}} + \frac{\partial\hat{n}}{\partial\mathbf{p}}\frac{\partial\varepsilon_1}{\partial\mathbf{r}}\right)\\ + e\mathbf{E}\frac{\partial\hat{n}}{\partial\mathbf{p}} + \frac{e}{2c}\left\{\left[\mathbf{v}\times\mathbf{H}\right]\frac{\partial\hat{n}}{\partial\mathbf{p}} + \frac{\partial n}{\partial\mathbf{p}}\left[\mathbf{v}\times\mathbf{H}\right]\right\} + \frac{\beta}{2\hbar}\left(\hat{\boldsymbol{\sigma}}\mathbf{H}\hat{n} - \hat{n}\hat{\boldsymbol{\sigma}}\mathbf{H}\right) = I(\hat{n}).\end{gathered} \tag{14}$$

Here $\hat{n}(\mathbf{p}, \mathbf{r})$ is the distribution function, which is a function of the coordinates and of the momenta and which is a matrix in spin space, σ is the operator of electron spin, β is the magnetic moment of the electron, $I(\hat{n})$ is the collision integral, and $\delta\epsilon_1$ differs from expression (12) by the absence of self-consistent term,

$$\delta\varepsilon_1 = \mathrm{Sp}_{\sigma'}\int\hat{f}(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}', \mathbf{r})\,d\mathbf{p}' - \frac{1}{2}\beta\hat{\sigma}\mathbf{H}. \tag{15}$$

In Eq. (14) and also Eq. (13) the value of $\mathbf{v}$, the velocity of the particles, in fact remains undetermined. If we neglect the effect of the lattice and consider an isotropic electron liquid, then it is possible,

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*Such an approach, applied to electron plasma, is developed in the work of Vlasov;[5] for metal electrons this approach is extensively used in the theory of the anomalous skin effect.[6]

according to Landau, to use the Gallilean relativity principle. Then

$$\mathbf{v} = \frac{\partial \varepsilon}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} + \mathrm{Sp}_{\sigma'} \int \hat{f}(\mathbf{p}, \mathbf{p}') \frac{\partial \hat{n}}{\partial \mathbf{p}'} d\mathbf{p}', \tag{16}$$

where m is the mass of the free electron. However, in a real metal m may differ from the mass of the free electron. Furthermore, this quantity, as well as the function $\hat{f}(\mathbf{p}, \mathbf{p}')$, can depend in principle on the direction. In addition, the region of large electromagnetic fields, $\hat{f}(\mathbf{p}, \mathbf{p}')$ may also depend on the electromagnetic field.

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[1] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956), Soviet Phys. JETP **3**, 920 (1956).

[2] D. Pines, Solid State Physics, Vol. 1 (Academic Press, N. Y. 1955), p. 367.

[3] L. D. Landau and I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **7**, 379 (1937); V. L. Ginzburg and V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **29**, 64 (1955), Soviet Phys. JETP **2**, 46 (1956).

[4] V. P. Silin, Физика металлов и металловедение (Physics of Metals and Metal Research) **3**, 193 (1956).

[5] A. A. Vlasov, Уч. зап. МГУ (Scientific Notes, Moscow State Univ.) **75**, vol. 2, part 1.

[6] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. **195**, 336 (1949).

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APPLICATION OF MATRIX POLYNOMIALS TO DETERMINE SCATTERING PHASES

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A formalism of invariant matrix polynomials $L^{s',s}_{\ell',\ell;J}$ is developed for systems of particles of arbitrary spin. A general method for calculating $L^{s',s}_{\ell',\ell;J}$ is found, and the polynomials for total spin 0, ½ and 1 are written out explicitly. Equations (3.2) — (3.7) make the expansion of any invariant operator in polynomials $L^{s',s}_{\ell',\ell;J}$ a simple matter. It is shown that the coefficients in such an expansion of the scattering matrix are directly related to the phase-shifts. Formulae are derived for calculating the phase-shifts to any order of perturbation theory. In many specific examples this method simplifies the calculation of phase-shifts.

1. INTRODUCTION

To make comparisons of various theories with experiment, one must deal either with cross-sections or with phase-shifts. Whenever possible the phase-shifts are to be preferred, since they express the properties of the scattering with maximum conciseness. For example, the absolute sign of a phase is highly significant, as it indicates a qualitative difference (attractive or repulsive interaction) between two processes which may have equal cross-sections.

There is no existing theory which describes satisfactorily the strong interactions. But there are several theories[1-3] which give a reasonable qualitative picture of some particular processes. To compare these theories with one another, it is also convenient to study the behavior of the phase-shifts which they predict.