ENERGY DISTRIBUTIONS IN TWO-PARTICLE DECAYS

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Isotropic decay into two particles is treated in the laboratory reference system. It is shown that the distribution of the secondaries with respect to the quantity v defined by Eq. (42) has a number of simple properties ("symmetry properties") which are independent of the energy distribution of the primaries. Formulas are given which enable one, from the v-spectrum of one of the secondaries, to determine the v-spectra of the other secondary and of the primary particle. Symmetry properties are also found for the spectra of the products of a cascade decay. Similar formulas for the energy spectra can be obtained by a simple computation. When the angular distribution is taken into account, these formulas are approximately applicable to the high energy part of the spectrum. In addition to quantitative applications, it is pointed out that the relations derived here may be used for identification of the mass of the primaries and for establishing the isotropy or the cascade nature of the decay.

In the two-particle breakup of an elementary particle, the energy spectrum of the decay products is determined by the kinematic features of the decay and the energy spectrum of the primary. The simplicity of this relation results in a number of general properties of the spectra of secondary particles in the laboratory reference system. For decays into two photons, these properties were explained by Carlson et al.[1] Subsequently, Sternheimer[2] solved the problem of determining the high energy part of the spectrum of the parimary particle (including its angular distribution) from the spectrum of its (arbitrary) decay products. In the present paper, after simplifying the computational procedure, we explain the internal properties of the spectrum of the secondaries. We then solve the problem of finding the energy spectrum and the mass of the primary particle from the energy spectrum of the secondary particle, as well as the problem of finding the spectrum and mass of one of the secondaries from the spectrum of the other.

1. PRIMARY FORMULA

Suppose that in a certain volume there occurs the reaction $A \to a_1 + a_2$ of decay of the particle A having mass M, momentum P and total energy E, into two particles characterized by m_1, m_2, p_1, p_2, e_1, e_2. We shall use an asterisk to denote these same quantities in the center of mass system (c.m.s.). Only the energy e_1 of the particle a_1 is recorded, without regard to its direction of motion. The following assumptions are made: the decay of A occurs isotropically; the energy distribution of A is given in the energy interval from $E_{min} = M$ to E_{max}; all the particles a_1 are recorded; the statistics are sufficiently good so that the shape of the energy distribution is precisely known.

The distribution of A in energy and angle is given by $\overline{N}(E, \vartheta)\, dE\, d\cos\vartheta$. We know that, for the case of a two-particle decay with isotropic distribution of the decay products in the c.m.s., the energy distribution of the secondaries for a fixed energy of the primary is equal to the constant $(M/2Pp^*)de_1$, if e_1 satisfies the inequality $e_{1\,min} \le e_1 \le e_{1\,max}$, and is equal to zero outside of the interval $(e_{1\,min}, e_{1\,max})$. The energy e_1 reaches its largest (smallest) value when the particle a emerges forward (backward) relative to the motion of particle A. Lorentz transformations give

$$\begin{matrix} e_{1\,min} \\ e_{1\,max} \end{matrix} = \frac{Ee_1^* \mp Pp^*}{M}. \tag{1}$$

The number of particles with energy e_1 in the interval of width de_1, produced from particles A with energy in the interval dE and direction in $d\vartheta$, is

$$(M/2Pp^*)\overline{N}(E, \vartheta)\,dE\,d\cos\vartheta\,de_1,$$

while the number of particles a_1, formed from particles A with arbitrary ϑ, is

$$(M/2Pp^*)\widetilde{N}(E)\,dE\,de_1,$$

where $\widetilde{N}(E) = \int_\vartheta \overline{N}(E, \vartheta)\,d\cos\vartheta$ is the average of the energy spectrum of A over all directions. In addition, the number density of particles a_1 in the energy interval de_1, formed from the decay of particles A of arbitrary energy, will be

$$\widetilde{n}_1(e_1) = \int_{E_{\min}}^{E_{\max}} (M/2Pp^*)\widetilde{N}(E)\,dE, \quad (2)$$

where $E_{\min}$ and $E_{\max}$ are the smallest and largest energies of particles A which produce particles with energy e_1.

We proceed to simplify Eq. (2). To do this we shall characterize the motion of the particles by positive quantities u or v according to the formulas

$$E = M\cosh u, \quad e_{1,2} = m_{1,2}\cosh v_{1,2}. \quad (3)$$

It follows from (1) that under Lorentz transformations the arguments for particles a_1 which move forward (backward) are expressed simply as:

$$v_{1\max} = v_1^* + u \text{ (forward)}, \quad v_{1\min} = |\,v_1^* - u\,| \text{ (backward)}. \quad (4)$$

Consequently, as u increases from 0 to ∞, $v_{1\max}$ increases from v_1^* to ∞; $v_{1\min}$ decreases from v_1^* to 0 in the interval where u increases from 0 to v_1^*, and then increases from 0 to ∞ when u goes from v_1^* to ∞. From (4) it follows that particles a_1 with argument $v_1 \geq v_1^*$ arise only from particles A with arguments in the interval $(v_1 - v_1^*, v_1 + v_1^*)$, while particles a_1 with argment $v_1 \leq v_1^*$ come from particles A with arguments in the interval $(v_1^* - v_1, v_1 + v_1^*)$. Equation (2) can therefore be written as

$$n_1(v_1) = k\int_{|v_1 - v_1^*|}^{v_1 + v_1^*} N(u)\,du, \quad (5)$$

where

$$k = M/2p^*, \quad \text{a} \quad n_1(v) \equiv \widetilde{n}_1(m_1\cosh v), \quad N(u) \equiv \widetilde{N}(M\cosh u)$$

are the "densities" for the energy distributions of particles a_1 and A converted to the arguments v and u. It should be remembered that to get, say, the number of particles with arguments between α and β, $n_1(v)$ must be weighted by the factor $m_1 \sinh v$:

$$\int_\alpha^\beta m_1 \sinh v\, n_1(v)\,dv.$$

The transformations (3) are not applicable when the rest mass is zero. However, if we map the half-line $e_1 > 0$ onto the line v_1 by means of

$$e_1 = e_1^* \exp v_1, \quad (6)$$

then the Lorentz transformations take the form $v_{1\max} = u$ (forward), $v_{1\min} = -u$ (backward), and the relation into which (2) goes for $m_1 = 0$,

$$\widetilde{n}_1(e_1) = \int_{E_{\min}}^{\infty} (M/2Pp^*)\widetilde{N}(E)\,dE$$

becomes

$$n_1(v_1) = k \int_{|v_1|}^{\infty} N(u)\,du. \tag{7}$$

Here we have

$$n_1(v_1) \equiv \tilde{n}_1(e_1^* \exp v_1).$$

The simple form of (5), (7) makes them convenient for further calculations.

2. RELATION TO THE MASS OF THE PRIMARY PARTICLE

Let $N(u) \equiv 0$ for $u \geq U$, where $U \leq v_1^*$. Then the upper limit in (5) is U. Since the integrand $N(u)$ is non-negative, it follows from (5) that $n_1(v_1)$ is a monotonically decreasing function of the lower limit $|v_1 - v_1^*|$. Its maximum value is attained for $v_1 = v_1^*$, and $n_1(v_1) \leq n_1(v_1^*)$ for all $v_1 \neq v_1^*$. Thus we find property A: the spectrum of the secondary particle, in the case where the arguments of particle A do not exceed $U \leq v_1^*$ (i.e., its energy is no greater than $E^{(1)} = Me_1^*/m_1$), has a maximum (peak or plateau) at the point with abscissa $v_1 = v_1^*$. The spectrum of gamma quanta shows a similar property at $v_1 = 0$ for any energy range of the primary particles.*

Since n_1 depends only on $|v_1 - v_1^*|$, we have property B — the spectrum $n_1(v_1)$ is symmetric around the point v_1^*:

$$v_1^* = (v' + v'')/2, \tag{8}$$

where v' and v'' are the abscissas of points in the spectrum with equal ordinates.

Similarly, the spectrum of gamma quanta has the property

$$v' + v'' = 0. \tag{8'}$$

Properties A and B enable us to determine the mass of one of the particles if we know the masses of the other two in the reaction, since v_1^* is given in terms of the masses of the particles by

$$(M^2 + m_1^2 - m_2^2)/2M = m_1 \cosh v_1^*. \tag{9}$$

The expression for M has the form

$$M/m_1 = \cosh v_1^* + [\sinh^2 v_1^* + (m_2/m_1)^2]^{1/2}. \tag{10}$$

For $m_1 = m_2 = m$, Eq. (10) reduces to

$$M = 2m \cosh v_1^*. \tag{11}$$

The set (10) and (8) determine M either from the position of the peak or from the abscissas of the points on a horizontal line. Returning to the usual energy spectra gives

$$M = [(e_1'e_1'' + p_1'p_1'' + m_1^2)/2]^{1/2} + [e_1'e_1'' + p_1'p_1'' + 2m_2^2 - m_1^2)/2]^{1/2}; \tag{10'}$$

or, for $m_1 = m_2 = m$:

$$M = [2(e_1'e_1'' + p_1'p_1'' + m^2)]^{1/2}, \tag{11'}$$

which reduces, for $m = 0$, to the formula $M = 2(e_1'e_2')^{1/2}$ derived in Ref. 1. From (8′) or (10′) we can also get the formula for the case of $m_1 = 0$, $m_2 \neq 0$:

$$M = (e_1'e_1'')^{1/2} + (e_1'e_1'' + m_2^2)^{1/2}. \tag{12}$$

Formulas (10), (10′), (11), and (11′) are valid in the absence of primary particles with energies above $E^{(1)} = Me_1^*/m_1$. Later we shall weaken these restrictions. Formula (12) is valid without any restriction of this sort.

Property B might be called the logarithmic symmetry of the spectrum of secondary particles,[4] since the argument is given in terms of the logarithm

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*We note that the condition $u = v_1^*$ has a simple physical meaning: for values of u higher than v_1^*, the secondary particle can only move forward in the laboratory system, i.e., there is a limiting angle of emergence. For $u < v_1^*$ there is no limiting angle.

$$v_1 = \ln[(e_1 + p_1)/m_1] \qquad (m_1 \neq 0), \tag{13}$$

$$v_1 = \ln(e_1/e_1^*) \qquad (m_1 = 0). \tag{13'}$$

For non-relativistic energies, v_1 coincides with the velocity of the particle.

Property B can also be used to determine m_2 when m_1 and M are known:

$$m_2 = (M^2 + m_1^2 - 2Mm_1 \cosh v_1^*)^{1/2}. \tag{14}$$

3. DETERMINATION OF THE SPECTRUM OF THE PRIMARY PARTICLE

We differentiate (5) with respect to v_1 for $v_1 \geq v_1^*$:

$$k^{-1}n_1'(v_1) = N(v_1 + v_1^*) - N(v_1 - v_1^*).$$

To solve this difference equation for N, we set $v_1 - v_1^* = u$ and give u successive values $u + 2v_1^*$, $u + 4v_1^*$, ... Summing these equations gives†

$$N(u) = -k^{-1} \sum_{\nu=0}^{\nu'} \tilde{n}_1' [u + (2\nu + 1) v_1^*]. \tag{15}$$

The summation ends when $n_1' = 0$. Thus the number density of primary particles with argument u is proportional to the sum of the derivatives of the spectrum of secondaries at points with abscissas forming an arithmetical progression with difference $2v_1^*$ and initial term $u + v_1^*$. In the usual notation, (15) has the form

$$\tilde{N}(E) = -(2p^*/M) \sum_{\lambda} (\varepsilon_\lambda P + \rho_\lambda E) \tilde{n}_1' (\varepsilon_\lambda E + \rho_\lambda P), \tag{15'}$$

where

$$\varepsilon_\lambda = \frac{m_1}{M} \cosh \lambda v_1^*, \quad \rho_\lambda = \frac{m_1}{M} \sinh \lambda v_1^*, \lambda = 1, 3, 5, \ldots, \tag{16}$$

while v_1^* is given by Eq. (9).

The high energy part of the spectrum (where $E \simeq P$) can be determined from the equation

$$\tilde{N}(E) \simeq -(2m_1 p^*/M^2) E \sum_{\lambda} \exp(\lambda v_1^*) \tilde{n}_1' (E \exp \lambda v_1^*). \tag{17}$$

As an example, we consider the reaction $\theta^0 \to 2\pi$. Here $v_\pi^* = 1.18$, so that

$$\tilde{N}(E) = -0.829 \{(0.5P + 0.415E) \tilde{n}' (0.5E + 0.415P) + 4.84 (P + E) \tilde{n}' [4.84 (P + E)] + \cdots\},$$

and for large E,

$$\tilde{N}(E) \simeq -0.829 E [0.915 \tilde{n}' (0.915E) + 9.69 \tilde{n}' (9.69E) + \ldots]. \tag{18}$$

The last formula coincides with (42a) of Ref. 2, although it was derived under different assumptions. Later we shall explain the reason for this coincidence.

For the case of $m_1 = 0$, the formula for N was derived in Ref. 1. In our notation it has the form

$$N(u) = -k^{-1} n_1'(u) \tag{19}$$

and is valid even for $m_2 \neq 0$.

4. THE SPECTRUM OF THE OTHER SECONDARY PARTICLE

By substituting in the expression for the spectrum of a_2 ($m_2 \neq 0$):

$$n_2(v_2) = k \int_{|v_2 - v_2^*|}^{v_2 + v_2^*} N(u)\, du$$

†The prime denotes differentiation with respect to the argument.

the spectrum of A as obtained from (15) or (19), we can construct the spectrum of a_2 from that of a_1:

$$n_2(v_2) = \sum_\lambda [n_1(|v_2 - v_2^*| + \lambda v_1^*) - n_1(v_2 + v_2^* + \lambda v_1^*)] \quad (m_1,\ m_2 \neq 0), \tag{20}$$

$$n_2(v_2) = n_1(|v_2 - v_2^*|) - n_1(v_2 + v_2^*) \quad (m_1 = 0,\ m_2 \neq 0). \tag{21}$$

(In all sums, $\lambda = 1, 3, 5, \ldots$)
If $m_2 = 0$, the spectrum of a_2 is expressible in terms of the spectrum of a_1 by means of (7) and (15):

$$n_2(v_2) = \sum_\lambda n_1(|v_2| + \lambda v_1^*) \quad (m_1 \neq 0,\ m_2 = 0). \tag{22}$$

Reverting to the usual parameters, energy and momentum, we rewrite (20)–(22) as

$$\begin{aligned} \tilde{n}_2(e_2) &= \sum_\lambda [\tilde{n}_1(\alpha_\lambda^+ e_2 - \beta_\lambda^+ p_2) - \tilde{n}_1(\alpha_\lambda^+ e_2 + \beta_\lambda^+ p_2)] \quad (e_2 \leqslant e_2^*), \\ \tilde{n}_2(e_2) &= \sum_\lambda [\tilde{n}_1(\alpha_\lambda^- e_2 + \beta_\lambda^- p_2) - \tilde{n}_1(\alpha_\lambda^+ e_2 + \beta_\lambda^+ p_2)] \quad (e_2 \geqslant e_2^*), \end{aligned} \tag{20'}$$

where m_1, $m_2 \neq 0$, and

$$\begin{matrix}\alpha_\lambda^\pm \\ \beta_\lambda^\pm\end{matrix} = \frac{m_1}{m_2} \begin{matrix}\cosh \\ \sinh\end{matrix}(\pm v_2^* + \lambda v_1^*). \tag{23}$$

For $m_1 = 0$, $m_2 \neq 0$:

$$\begin{aligned} \tilde{n}_2(e_2) &= \tilde{n}_1[e_1^* M(e_2 - p_2)/m_2^2] - \tilde{n}_1[e_1^* M(e_2 + p_2)/m_2^2] \quad (e_2 \leqslant e_2^*), \\ \tilde{n}_2(e_2) &= \tilde{n}_1[e_1^*(e_2 + p_2)/M] - \tilde{n}_1[e_1^* M(e_2 + p_2)/m_2^2] \quad (e_2 \geqslant e_2^*). \end{aligned} \tag{21'}$$

For $m_1 \neq 0$, $m_2 = 0$:

$$\begin{aligned} \tilde{n}_2(e_2) &= \sum_\lambda \tilde{n}_1[(m_1 e^{-\lambda v_1^*} e_2/2e_2^*) + (m_1 e^{\lambda v_1^*} e_2^*/2e_2)] \quad (e_2 \leqslant e_2^*), \\ \tilde{n}_2(e_2) &= \sum_\lambda \tilde{n}_1[(m_1 e^{\lambda v_1^*} e_2/2e_2^*) + (m_1 e^{-\lambda v_1^*} e_2^*/2e_2)] \quad (e_2 \geqslant e_2^*). \end{aligned} \tag{22'}$$

We give some particular cases of these formulas. For the reaction $\Xi^- \to \Lambda^0 + \pi^-$, $e_\Lambda \geq 1.1192$ Bev,

$$n_\Lambda(e_\Lambda) = n_\pi(0.17\,e_\Lambda + 0.11\,p_\Lambda) - n_\pi(0.20\,e_\Lambda + 0.15\,p_\Lambda) + n_\pi(0.81\,e_\Lambda + 0.80\,p_\Lambda) - n_\pi(1.05\,e_\Lambda + 1.04\,p_\Lambda) + \ldots$$

For the reaction $\Sigma^0 \to \Lambda^0 + \gamma$:

$$n_\Lambda(e_\Lambda) = n_\gamma(0.065\,(e_\Lambda + p_\Lambda)) - n_\gamma(0.075\,(e_\Lambda + p_\Lambda)) \quad (e_\Lambda \geqslant 1.1126 \text{ Bev}),$$

$$n_\gamma(e_\gamma) = n_\Lambda(6.7\,e_\gamma + 0.046\,e_\gamma^{-1}) + n_\Lambda(5.8\,e_\gamma + 0.053\,e_\gamma^{-1}) + \ldots \quad (e_\gamma \leqslant 0.0773 \text{ Bev}),$$

$$n_\gamma(e_\gamma) = n_\Lambda(7.7\,e_\gamma + 0.040\,e_\gamma^{-1}) + n_\Lambda(8.9\,e_\gamma + 0.035\,e_\gamma^{-1}) + \ldots \quad (e_\gamma \geqslant 0.0773 \text{ Bev}).$$

5. INTERNAL PROPERTIES OF THE SPECTRA OF SECONDARY PARTICLES

Equation (15) shows that the spectrum of A can be determined from that part of the spectrum of a_1 for which $v_1 \geq v_1^*$. On the other hand, the spectrum of A determines the whole spectrum of a_1, including the region $v_1 \leq v_1^*$. There must therefore be a relation between the regions $v_1 \leq v^*$ and $v_1 \geq v^*$.† In fact, substituting (15) in (5) and setting $v_1 \leq v_1^*$, we get (property C):

$$n_1(v_1) = n_1(2v_1^* - v_1) - n_1(2v_1^* + v_1) + n_1(4v_1^* - v_1) - n_1(4v_1^* + v_1) + \ldots \tag{24}$$

In Sec. 2 we presented a method for determining v_1^* and indicated the limiting energy up to which this procedure was applicable. Formula (24) enables us, at least in principle, to reduce and finally to eliminate this restriction on the energy of the primary particle.

†M. I. Podgoretskii called my attention to this point.

The arguments of the successive terms on the right side of (24) vary respectively within the intervals $(v_1^*, 2v_1^*)$, $(2v_1^*, 3v_1^*)$, etc. If $U \le v_1^*$, then $v_{1\,max} \le 2v_1^*$, and all terms in (24) except the first vanish, so that we are left with the equation

$$n_1(v_1) = n_1(2v_1^* - v_1) \tag{25}$$

which is a statement of property B (cf. above). Next, let $v_1^* \le U < 2v_1^*$. Then $v_{1\,max} = U + v_1^* < 3v_1^*$, and we must take two terms on the right of (24). However, for sufficiently large v_1 the second term is identically zero. This begins when the inequality $2v_1^* + v_1 \ge U + v_1^*$ is satisfied, so that $v_1 \ge U - v_1^*$, and the argument of the first term on the right is $3v_1^* - U$. We finally get the following result: if $v_1^* \le U < 2v_1^*$, then (25) is valid in the neighborhood of v_1^*, in the interval $(U - v_1^*, 3v_1^* - U)$; outside of this interval, in the neighborhood of 0 and $2v_1^*$, i.e., in the intervals $(0, U - v_1^*)$, $(3v_1^* - U, U + v_1^*)$, we have the equation

$$n_1(v_1) = n_1(2v_1^* - v_1) - n_1(2v_1^* + v_1). \tag{26}$$

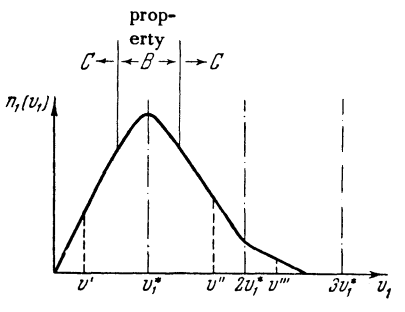

FIG. 1

Having found three points v', v'', v''' such that (26) is satisfied [i.e., $n_1(v') = n_1(v'') - n_1(v''')$] and for which $2v' = v''' - v''$, we determine $v_1^* = (v' + v'')/2$ (property C; see Fig. 1).

Similarly, we can show that if $2v_1^* \le U < 3v_1^*$, then in the neighborhoods of 0 and $2v_1^*$, in the intervals $(0, 3v_1^* - U)$, $(U - v_1^*, 5v_1^* - U)$, Eq. (26) is satisfied, while outside, in the neighborhood of v_1^* and $3v_1^*$, i.e., in the intervals $(3v_1^* - U, U - v_1^*)$, $(5v_1^* - U, U + v_1^*)$, the condition is

$$n_1(v_1) = n_1(2v_1^* - v_1) - n_1(2v_1^* + v_1) + n_1(4v_1^* - v_1) \tag{27}$$

etc.

Thus, from $U \sim 0$ up to $U^{(2)} = 2v_1^*$ there is a portion of the spectrum where property B manifests itself, while from $U^{(1)} = v_1^*$ up to $U^{(3)} = 3v_1^*$ there is a part of the spectrum showing property C′, etc. The limitation on the energy of the primary particle which was given in Sec. 1 is thus removed.

In the table, we give the limiting values of the energy, $E^{(1)}$, $E^{(2)}$, $E^{(3)}$ corresponding to $U^{(1)}$, $U^{(2)}$, $U^{(3)}$ for several decay reactions. For light secondaries $(m_1 \ll M)$ these limits are exceedingly high and go far out in the relativistic region.

Limiting Energies of Primary Particle (Bev)

Reaction	Spectrum	$E^{(1)}$	$E^{(2)}$	$E^{(3)}$
$\pi \to \mu + \nu$	μ	0.146	0.16	0.19
$\theta^0 \to 2\pi$	π	0.85	2.5	7.7
$\Lambda^0 \to p + \pi^-$	p	1.115	1.13	1.16
$\Lambda^0 \to p + \pi^-$	π	1.33	2.1	3.6
$\Sigma^+ \to p + \pi^0$	π	1.97	5.3	16
$\Xi^- \to \Lambda^0 + \pi^-$	π	1.88	4.0	9.6
$\Sigma^0 \to \Lambda^0 + \gamma$	γ	∞	—	—

The spectrum of a_1 has an integral property (property D). Integrating (24) between the limits $v_1 = 0$ and $v_1 = v_1^*$, and making a change of variables of the form $2\nu v_1^* \pm v_1 = t$ in each of the integrals, we get

$$\int_0^{v_1^*} n_1(v_1)\,dv_1 - \int_{v_1^*}^{2v_1^*} n_1(v_1)\,dv_1 + \int_{2v_1^*}^{3v_1^*} n_1(v_1)\,dv_1 - \ldots = 0, \tag{28}$$

i.e., if we break up the spectrum of the secondary particle into pieces of length v_1^* (where the last piece may have a shorter length), then the sum of the areas of the even-numbered pieces is equal to the sum of the areas of the odd-numbered pieces. By finding a length v, such that property D is satisfied, we determine $v_1^* = v$ and M to an accuracy exceeding that attainable by using properties A to C.

Formula (28) can also be expressed in the usual notation as

$$\Big[\int_0^{e_1^*} - \int_{e_1(v_1^*)}^{e_1(2v_1^*)} + \int_{e_1(2v_1^*)}^{e_1(3v_1^*)} - \ldots\Big] \frac{\tilde{n}_1(e_1)}{p_1}\,de_1 = 0, \tag{29}$$

where $e_1(x) = m_1 \cosh x$.

6. THE SPECTRUM OF THE PRODUCTS OF A CASCADE DECAY

Let us consider the case where one of the products of a decay $A \to a_1 + a_2$ in turn decays into two particles, $a_1 \to \alpha_1 + \alpha_2$. Examples of this are the cascades

$$\Xi^- \to \pi^- + \Lambda^0 \to \pi^- + (\pi^- + p) \text{ or } \to \pi^- + (\pi^0 + n), \quad \Xi^0 (\Sigma^0) \to \Lambda^0 + \pi^0 \to \Lambda^0 + 2\gamma, \ \Lambda^0 \to n + \pi^0 \to n + 2\gamma.$$

In the last three cascades, for low energies, the two decay processes occur practically at one point. The spectrum of a_1 has axial symmetry, so that the spectrum of α_1 should have some additional symmetry. In order to reveal this additional property of the energy spectrum of the products of a cascade decay, let us assign to the particle α_1 the argument w with the characteristic value w^*, and assume first that the mass of α_1 is not equal to zero. The distribution of α_1 with respect to the argument w will be

$$\nu(w) = \int\limits_{|w-w^*|}^{w+w^*} k_2 dv \int\limits_{|v-v_1^*|}^{v+v_1^*} k_1 N(u)\, du, \tag{30}$$

where

$$k_2 = m_1 / 2 m_{\alpha_1} \sinh w^*, \quad k_1 = M / 2p^*.$$

Let us assume that $v_1^* < w^*$. This condition is satisfied for all the cascade processes cited above. Also, let us first assume that $w \geq w^* + v_1^*$. Then the absolute value signs in (30) can be dropped. Differentiating (30) twice, we get:

$$(k_2 k_1)^{-1} \nu''(w) = N(w + w^* + v_1^*) - N(w + w^* - v_1^*) - N(w - w^* + v_1^*) + N(w - w^* - v_1^*).$$

The procedure for solving this difference equation is a repetition of the computations we did earlier, and leads to

$$N(u) = (k_1 k_2)^{-1} \sum_{m=0}^{m'} \sum_{n=0}^{n'} \nu''(u + k_n w^* + l_m v_1^*), \quad k_n = 2^{n+1} - 1, \ l_m = 2^{m+1} - 1; \ m, n = 0,1,2,\ldots \tag{31}$$

Now that we have determined the spectrum of A from the piece $w \geq w^* + v_1^*$ of the spectrum of α_1 by using (31), we can relate other portions of the spectrum of α_1 to this one. Setting $k_n w^* + \ell_m v_1^* = \lambda$ and and substituting (31) into (30), we get:

$$\nu(w) = \sum_{\nu} \int\limits_{|w-w^*|}^{w+w^*} dv_1 \int\limits_{|v_1-v_1^*|}^{v_1+v_1^*} \nu''(u+\lambda)\, du. \tag{32}$$

First let $w^* \leq w \leq w^* + v_1^*$. Then the computations give

$$\nu(w) = \sum_{\lambda} [2\nu(\lambda) + \nu(w + w^* + v_1^* + \lambda) - \nu(w - w^* + v_1^* + \lambda) - \nu(w + w^* - v_1^* + \lambda) - \nu(-w + w^* + v_1^* + \lambda)]. \tag{33}$$

Next, in the interval $w^* - v_1^* \leq w \leq w^*$ Eq. (33) is again satisfied. Finally, for $w \leq w^* - v_1^*$,

$$\nu(w) = \sum_{\lambda} \nu[(w + w^* + v_1^* + \lambda) - \nu(w - v^* + v_1^* + \lambda) - \nu(-w + w^* + v_1^* + \lambda) + \nu(-w - w^* + v_1^* + \lambda)].$$

The complexity of the formulas is an obstacle to a demonstration of the symmetry of the spectrum. However, if the spectrum of A cuts off at $U \leq v_1^*$, and as a consequence of this the spectrum of α_1 is cut off at $U + w^* + v_1^* \leq w^* + 2v_1^*$, then in (31) all that remains of the sum is the first term $\nu''(u + w^* + v_1^*)$, while in (33) we are left with $\nu(w) = 2\nu(w^* + v_1^*) - \nu(-w + 2w^* + 2v_1^*)$, which can be written in the form

$$[\nu(w) + \nu(2w^* + 2v_1^* - w)] / 2 = \nu(w^* + v_1^*) \tag{34}$$

which shows that the pieces of the spectrum in the intervals $(w^*, w^* + v_1^*)$ and $(w^* + v_1^*, w^* + 2v_1^*)$ are transformed into one another under inversion in the point $(w^* + v_1^*, \nu(w^* + v_1^*))$ (see Fig. 2). If in addition $2v_1^* \leq w^*$, then $v_{1\,max} < w^*$, and the spectrum of α_1 should also possess axial symmetry with respect to the line $w = w^*$. (The image is shown as a dotted curve in Fig. 2.) As we extend the range of the arguments, the connection between the parts of the spectrum becomes complicated.

We must also consider the case where the mass of α_1 is equal to zero. Considerations entirely analogous to those used above give

$$N(u) = (k_1 k_2)^{-1} \sum_n \nu''(u + (2n+1) v_1^*),$$

from which we get the following connection between the parts of the spectrum: for $-v_1^* \le w \le v_1^*$,

$$\nu(w) = [2\nu(v_1^*) - \nu(-w + 2v_1^*) - \nu(w + 2v_1^*)] + [2\nu(3v_1^*) - \nu(-w + 4v_1^*) - \nu(w + 4v_1^*)] + \ldots; \tag{35}$$

for $w \le -v_1^*$:

$$\nu(w) = \nu(-w).$$

If, as was assumed above, the spectrum of A ends at $U \le v_1^*$, then (35) reduces to

$$\nu(w) = 2\nu(v_1^*) - \nu(2v_1^* - w), \tag{36}$$

i.e., the point $(v_1^*, \nu(v_1^*))$ is a center of symmetry for the portions of the spectrum in the intervals $(0, v_1^*)$, $(v_1^*, 2v_1^*)$. It is easy to see that the point $(-v_1^*, \nu(-v_1^*))$ also has this property (see Fig. 3).

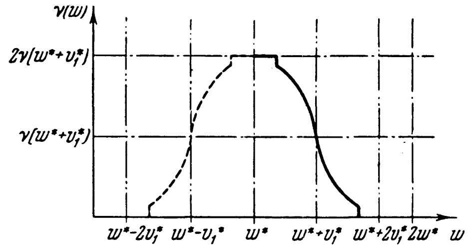

FIG. 2

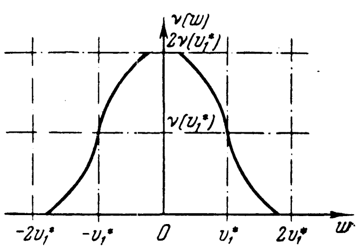

FIG. 3

If the maximum value U satisfies the condition $v_1^* \le U \le 2v_1^*$, the central symmetry disappears and (36) becomes more complicated:

$$\nu(w) = 2\nu(v_1^*) - \nu(2v_1^* - w) - \nu(2v_1^* + w).$$

The properties we have found may make it possible to distinguish the cascade decay $A \to a_1 + a_2 \to (\alpha_1 + \alpha_2) + a_2$ from the three-particle decay $A \to \alpha_1 + \alpha_2 + a_2$. We may expect that in the latter the spectrum of α_1 will not posses symmetry because of the anisotropy of the decay. From the location of the center of symmetry of the spectrum we can also estimate the mass of the primary particle.

Now let us consider the reaction $\pi^- + p \to \Sigma^0 + K^0 \to (\Lambda^0 + \gamma) + K^0$. The directions of Σ^0 and K^0 will not be distributed isotropically in the c.m.s. so that we should not expect any symmetry in the spectra of Σ^0 and K^0. However, the decay $\Sigma^0 \to \Lambda^0 + \gamma$ can occur isotropically, so that the spectra of Λ^0 and γ have the usual symmetry properties A to D of one-stage decay processes. The detection of such properties in the spectra of Λ^0 and γ may serve as an indication of the cascade character of the process. In addition, if the spectrum of Λ^0 is symmetric, we cannot admit any production of Λ^0 via the process $\pi^- + p \to \Lambda^0 + K^0$, which is anisotropic. Obviously, similar considerations can be applied to other reactions in which particles are created and which culminate in the decay of one of the produced particles.

7. EXTENSION OF VALIDITY OF FORMULAS

Throughout all of the preceding, the angular distributions of the secondary particles was not taken into account. It is possible to simplify Sternheimer's derivation[2] of the formulas relating the angular and energy spectra of the high energy primary and secondary particles. This will enable us to extend some of the results obtained above to the case where the secondary particle is observed in a definite direction.

Suppose that the particles a_1 all appear at the same point. Also let us assume that a definite direction of motion ϑ' is selected, and that we record the energies of particles a_1 moving in this direction. We take this direction as the polar axis. We shall assume that the distribution is uniform in azimuth. The probability for particle a_1 to have a direction in $d\vartheta'$ is

$$\frac{M}{2Pp^*} \frac{de_1}{d\cos(\vartheta - \vartheta')} d\cos\vartheta'.$$

Therefore the total number of particles moving in the direction $\vartheta' = 0$ will be

$$d\cos\vartheta' \int_E \int_\vartheta \overline{N}(E, \vartheta)\, dE\, d\cos\vartheta \,\frac{M}{2Pp^*}\, \frac{de_1}{d\cos\vartheta}\,.$$

If we require that $\cos\vartheta'$ and e_1 lie in the intervals $(1, 1 + d\cos\vartheta')$ and $(e_1, e_1 + de_1)$, then we must either treat E as a function of e_1 and ϑ or ϑ as a function of e_1 and E. The first assumption was made in Ref. 2. The number density of particles then takes the form

$$\widetilde{n}_1(e_1, 0) = \int_\vartheta \overline{N}\,[E(e_1, \vartheta), \vartheta]\, \frac{dE(e_1, \vartheta)}{de_1}\, \frac{M}{2Pp^*}\, \frac{de_1(\vartheta)}{d\cos\vartheta}\, d\cos\vartheta,$$

which necessitates further messy calculations. But on the second assumption, $\vartheta = \vartheta(E, e_1)$, we immediately get

$$\widetilde{n}_1(e_1 0) = (M/2p^*) \int_{E_{\min}}^{E_{\max}} \overline{N}\,[E, \vartheta(E, e_1)]\, p^{-1}\, dE.$$

This formula is essentially the same as (34) of Ref. 2, but is exact. Transforming to arguments u, v_1 in place of E, e_1, we can express the distribution with respect to u and v_1 as

$$n_1(v_1, 0) = k \int_{|v_1 - v_1^*|}^{v_1 + v_1^*} N(u, \vartheta(u, v_1))\, du, \tag{37}$$

where

$$\vartheta(u, v_1) = \arccos\,[(\cosh u \cosh v_1 - \cosh v_1^*)/\sinh u \sinh v_1]; \qquad n_1(v_1, 0) \equiv \widetilde{n}_1(m_1 \cosh v_1, 0);\ N(u, \vartheta) \equiv \overline{N}(M \cosh u, \vartheta).$$

For high energies (large values of u and v_1), the angle of emergence ϑ is close to zero, and (37) can be replaced by the approximate equation

$$n_1(v_1, 0) = k \int_{|v_1 - v_1^*|}^{v_1 + v_1^*} N(u, 0)\, du. \tag{38}$$

This equation is identical with (5). Therefore all the consequences of formula (5) which are not dependent on the condition that the energy be low are also valid for (38). So formulas (15), (17), (20) — (22), (20′) — (22′) are also valid for the case where the secondary particles are observed at a definite angle. Now they give the spectra of A or a_1 in the direction of observation, and not their spectra averaged over all angles. This explains why Eq. (18) of this paper coincides with (42a) of Ref. 2.

We present formulas, valid for high energies, which give the spectrum of one of the secondaries in terms of the spectrum of the other:

$$\widetilde{n}_2(e_2) = \sum_\lambda\, [\widetilde{n}_1(e_2 e^{-v_2^* - \lambda v_1^*}) - \widetilde{n}_1(e_2 e^{v_2^* + \lambda v_1^*})] \quad (m_1, m_2 \neq 0); \tag{39}$$

$$\widetilde{n}_2(e_2) = \widetilde{n}_1(2 e_1^* e_2/M) - \widetilde{n}_1(2 e_1^* M e_2/m_2^2) \quad (m_1 = 0, m_2 \neq 0), \tag{40}$$

$$\widetilde{n}_2(e_2) = \sum_\lambda \widetilde{n}_1(m_1 e^{\lambda v_1^*} e_2/2 e_2^*) \quad (m_1 \neq 0 \quad m_2 = 0). \tag{41}$$

These formulas are applicable when the limiting angles of emergence of both secondary particles are small.

8. CONCLUDING REMARKS

Thus, in a two-particle decay, to make clear the connection between the energy spectrum of the decay products and the energy spectrum of the primary component (averaged over all angles), it is convenient* to introduce the parameter

$$v = \cosh^{-1}\gamma = \cosh^{-1}(e/m) = \ln\frac{e+p}{m}. \quad (42)$$

The distribution of the decay products with respect to this parameter (the "v-spectrum") has simple properties which are independent of the energy distribution of the primary particles: (A) In a certain range of energies of the primary particles, the distribution of the secondaries has a maximum at a point v^* which is determined by the masses of the particles. (B) In this same energy range, the distribution of secondary particles is symmetric with respect to the vertical line through the point of the maximum. (C) When the energy range is extended, we retain the simple relation (24) between the points of the spectrum in even $(2\nu v^*, 2\nu v^* + v^*)$ and odd $(2\nu v^* + v^*, 2\nu v^* + 2v^*)$ intervals of the v-spectrum. (D) The total area under the v-spectrum in the even sections is equal to the total area of the odd sections.

In principle, the use of properties A to D enables us to identify the mass of the primary particles. These properties of v-spectra occur under the following assumptions: presence of only two decay products; isotropy of the decay in the center of mass system; absence of competing decay processes; limitation of the range of energies of the primary particles; absence of systematic errors in counting of secondary particles; absence of any preferred direction of observation; good statistics. We shall assume that those assumptions are fulfilled which depend on the arrangement of the experiment, and limit ourselves to the first two assumptions. Nothing in this paper shows that these are necessary conditions for the symmetry of the v-spectra. However, from physical considerations, it does follow with a definite probability. We can assign this same degree of reliability when (subject to the fulfillment of the other conditions) we apply the symmetry criteria to prove the isotropy or cascade character of a decay (as was recommended in Sec. 6).

The formulas for determining the energy spectra of the particles are suitable for use in experiments in emulsions or chambers where it is possible to avoid selection of a definite direction of observation. However, the existence[3-5] for certain angular distributions of "isotropic" directions, along which the flux of particles is the same as for an isotropic distribution, enables us to use these formulas in counter experiments also.

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Translated by M. Hamermesh
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*Cf. also Ref. 6.