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The Surface Impedance of Metals in the Infrared Region

M. I. KAGANOV AND V. V. SLEZOV
Physico-Technical Institute, Ukranian Academy of Sciences

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An expression has been obtained for the surface impedance of metals in the infrared region without making any special assumptions concerning the law of dispersion of the conduction electrons.

1. IN THE OPTICS OF METALS the most interesting region, it seems to us, is the infrared region in which the frequency of the electromagnetic field satisfies the condition

$$\nu_0 \ll \omega \ll \omega_a \qquad (1)$$

Here $\nu_0 = 1/\tau$ is the collision frequency (τ is the relaxation time), and ω_a is the limit of the internal photoeffect.* As shown in Ref. 1, for low temperatures and for pure metals this region is known to be essentially: $\nu_0 \sim 10^{11}$, $\omega_a \sim 10^{15}$ to 10^{16}. Thus we talk of working at wavelengths on the order of ten microns.

In this frequency region the electron gas is approximately described by the dielectric constant

$$\varepsilon = 1 - 4\pi Ne^2/m\omega^2 \qquad (2)$$

where N is the density of free electrons and m the effective mass.* In other words, the electron gas "behaves" like an electron plasma.

Since $\Omega^2 \approx 4\pi Ne^2/m \sim \omega_a^2$, then in the frequency region of interest to us the dielectric constant of the metal is negative ($\varepsilon < 0$), and, its modulus is considerably greater than 1, *i.e.*,

$$\varepsilon \approx -4\pi Ne^2/m\omega^2, \quad |\varepsilon| \gg 1.$$

The reflection of light from the surface of a metal in this case is, in principle, not connected with the ohmic loss and can be described by a purely imaginary surface impedance

* The internal photoeffect which is due to the interaction of electrons can occur at all frequencies; however, it principally occurs only for $\omega > \omega_a$ ($\hbar\omega_a$ is of the order of intervals between energy bands, $\sim 10^{-12}$ ergs).

* Note that according to the opinion of Ginzburg (*cf.* Ref. 1) we put in formula (2) the mass of the free electrons. All resultant changes in N/m are due to the change in the electron density N. This, of course, cannot lead to contradiction, but, it seems to us, causes trouble in comparing optically measured constants of the electron gas with results of other experiments (galvanomagnetic, specific heat, *etc.*). Moreover, if the calculation is carried out with the use of the kinetic equation, it is natural that in formula (2) there should appear precisely the effective mass (see below).

$$Z \approx iX = 4\pi i\omega / c\Omega \equiv i\sqrt{4\pi m\omega^2 / Ne^2c^2}. \quad (3)$$

However, formula (2), and consequently also (3), are approximate. Ohmic loss, naturally, occurs but $R = \mathrm{Re}\, Z \ll X$.

The real part R of the impedance is essentially different in two cases, occurring for frequencies satisfying inequality (1): normal skin effect ($\delta \gg l$; $l = v\tau$, $\delta = c/\Omega$):

$$R = 2\pi v_0 / c\Omega; \quad (4)$$

anomalous skin effect ($\delta \ll l$):

$$R = 3\pi v / 4c^2 \quad (5)$$

(v is the limiting Fermi velocity).

Formulas (3) to (5) were obtained on the basis of an isotropic quadratic dispersion relation ($\varepsilon = p^2/2m$; ε, p — energy and quasi-momentum of the electron) without allowing for quantum mechanical effects. The latter are due to the fact that $\hbar\omega$ for low temperatures not only is not less than kT, but may possibly be even considerably larger than kT. However, Wolfe[2] has shown that the results of quantum mechanical and classical calculations of ohmic loss connected with diffuse scattering of electrons on walls coincide. True, one must bear in mind (especially when comparing formulas of type (5) with experiment) that the surface loss is not always considerably larger than the volume loss.* For instance, according to Goldstein,[3] for $\hbar\omega \gg k\Theta$, kT (Θ is the Debye temperature) the quantum losses are of the same order as given by formula (5).

In the present article we obtain generalized formulas for the case of an arbitrary anisotropic dispersion law: $\varepsilon = \varepsilon(\mathbf{p})$. Just as in Refs. 1 and 4 to 8, the calculation is made with the aid of the classical kinetic equation.

2. Consider a plane monochromatic wave of frequency ω on a normally incident half-space $z > 0$ occupied by metal.

The complete system of equations describing the behavior of the electrons comprises the linearized kinetic equation

$$i\omega f + v_z \frac{\partial f}{\partial z} + \frac{f}{\tau} = -\frac{\partial f_0}{\partial \varepsilon} e\mathbf{E}\mathbf{v} \quad (6)$$

and the Maxwell equations

$$\mathrm{curl}\, \mathbf{H} = 4\pi \mathbf{j}/c, \mathrm{curl}\, \mathbf{E} = -\frac{i\omega}{c}\mathbf{H};$$
$$\mathbf{j} = \frac{2e}{(2\pi\hbar)^3}\int \mathbf{v} f\, d\mathbf{p}, \quad (7)$$

f is an addition to the equilibrium Fermi function and $v = \nabla_p \varepsilon$. The boundary condition for Eq. (6) serves as the condition for diffusion of the surface electron from the metal-vacuum boundary

$$f = 0 \quad \text{for} \quad v_z > 0, \; z = 0 \quad (8)$$

and for the vanishing of the non-equilibrium addition f inside the metal

$$f = 0 \quad \text{for} \quad z \to \infty. \quad (9)$$

We shall solve the systems (6) and (7) by successive approximations. Assuming ω considerably larger than v/δ and $1/\tau$ we find the zeroth approximation for the distribution function f and the electric field E, neglecting the second and third terms in Eq. (6). From the obtained field we determine the first approximate distribution function, a knowledge of which allows us to calculate the real part R of the impedance (the imaginary part X of the impedance is calculated from the zeroth approximation). Thus, the zeroth approximation is:

$$i\omega f = -(\partial f_0 / \partial \varepsilon)\, e\mathbf{E}\mathbf{v}.$$

Considering that $\partial f_0/\partial \varepsilon = -\delta(\varepsilon - \varepsilon_0)$ we have

$$j_i = \frac{2e^2}{(2\pi\hbar)^3\, i\omega} \oint v_i v_k \frac{dS}{v} E_k. \quad (10)$$

Integration is over the surface $\varepsilon(\mathbf{p}) = \varepsilon_0$ where ε_0 is the Fermi limiting energy, and dS an area element on the Fermi surface. From the fact that the current density and the electric field are out of phase by $\pi/2$, it is evident that we deal here not with the ohmic current but with a polarized electron gas.

We eliminate from Eq. (7) the magnetic field and, by means of the condition $j_z = 0$, the z component of the electric field.* Then we obtain an equation for the electric field

* If we do not take into account quantum mechanical effects, then in the case of the anomalous skin effect of interest to us $R_{\mathrm{surf}} \gg R_{\mathrm{vol}}$. Actually, according to (4) and (5): $R_{\mathrm{surf}}/R_{\mathrm{vol}} \sim v\Omega/cv_0 \sim l/\delta \gg 1$.

* Condition $j_z = 0$ is equivalent to the equation of continuity $\mathrm{div}\, j = 0$. We have omitted the term $\partial\rho/\partial t$ (ρ is the electron density), which corresponds to disregarding displacement current (*cf.* Ref. 7).

$$\frac{d^2E_\alpha}{dz^2} - \frac{4\pi e^2}{c^2}\left(\frac{N}{m}\right)_{\alpha\beta} E_\beta = 0, \tag{11}$$

where we introduce the designation

$$\left(\frac{N}{m}\right)_{\alpha\beta} = \frac{2}{(2\pi\hbar)^3}\oint\left\{\frac{v_\alpha v_\beta}{v}\,dS - \oint\frac{v_\alpha v_z}{v}\,dS\oint\frac{v_z v_\beta}{v}\,dS \Big/ \oint\frac{v_z^2}{v}\,dS\right\} \tag{12}$$

($\alpha, \beta = x, y$). In the isotropic (quadratic) case

$$d^2E_\alpha/dz^2 - 4\pi Ne^2E_\alpha/mc^2 = 0. \tag{13}$$

The last equation is obtained from Eq. (11) if we assume $\varepsilon = p^2/2m$, $v = p/m$ and take into account that $N = (8\pi/3)p^3(2\pi\hbar)^{-3}$. We align the x and y axes with the principal axes of the tensor $(N/m)_{\alpha\beta}$. Then

$$E_x = E_x(0)\,e^{-z/\delta_x};\quad E_y = E_y(0)\,e^{-z/\delta_y}, \tag{14}$$

where

$$\delta_x = c/\Omega_x;\quad \delta_y = c/\Omega_y;$$
$$\Omega_x^2 = 4\pi e^2(N/m)_x;\quad \Omega_y^2 = 4\pi e^2(N/m)_y \tag{15}$$

$(N/m)_x$, $(N/m)_y$ are the principal values of the tensor $(N/m)_{\alpha\beta}$. If the crystal is cubic then $\Omega_x = \Omega_y$.

The surface impedance in the zeroth approximation is pure imaginary

$$Z_x = 4\pi E_x(0)/cH_y(0) = 4\pi i\omega\delta_x/c^2 = 4\pi i\omega/c\Omega_x; \tag{16}$$

$$Z_y = -4\pi E_y(0)/cH_x(0) = 4\pi\omega i\delta_y/c^2 = 4\pi i\omega/c\Omega_y. \tag{17}$$

These formulas are natural generalizations of expression (3).

Note that if we do not take into account ohmic loss (to what degree this is valid will be shown further on), then we have two directions (axes x and y in our notation) in which the polarized reflected wave coincides with the incident one (we speak here of a plane polarized incident wave):

Assume for definiteness that a wave polarized longitudinal to the x axis falls on the metal. Then we have inside the metal a field (zeroth approximation):

$$E_x = E_x(0)\,e^{-z/\delta_x};\quad E_y = 0;\quad E_z = E_z(0)\,e^{-z/\delta_x};$$
$$E_z(0) = -E_x(0)\oint\frac{v_z v_x}{v}\,dS \Big/ \oint\frac{v_z^2}{v}\,dS. \tag{18}$$

The expression for E_z is obtained from the condition $j_z = 0$ (*cf.* above). Substituting expression (18) in Eq. (6) and integrating with allowance for boundary conditions (8) and (9) we find

$$f = -\frac{\partial f_0}{\partial\varepsilon}eE_x(0)\frac{v_x/v_z - \oint(v_z v_x/v)\,dS \Big/ \oint(v_z^2/v)\,dS}{(i\omega+\nu_0)/v_z - 1/\delta_x} \times\left\{\exp\left(-\frac{z}{\delta_x}\right) - A\exp\left(-\frac{i\omega+\nu_0}{v_z}z\right)\right\} \tag{19}$$

where

$$A = \begin{cases} 0, & \text{if } v_z < 0, \\ 1, & \text{if } v_z > 0. \end{cases}$$

Using (19) and the central symmetry of the Fermi surface, we calculate the total current flowing through the metal

$$J_\alpha = \int_0^\infty j_\alpha(z)\,dz = \frac{2e^2}{(2\pi\hbar)^3}E_x(0)\frac{\delta_x}{i\omega+\nu_0} \times\oint\left(v_x v_\alpha - v_\alpha v_z\oint\frac{v_z v_x}{v}\,dS \Big/ \oint\frac{v_z^2}{v}\,dS\right) \times\left(1+\frac{1}{1+(v_z/\delta_x)(i\omega+\nu_0)}\right)d/Sv. \tag{20}$$

Neglecting ν_0 compared with ω and $v_z/\delta_x\omega$ compared with unity, we obtain for the current a zeroth approximation which agrees with expression (10) if we substitute (18) for the field E_i. In particular, the y-component of the zeroth approximation for the current is equal to zero.

We are interested in the first approximation – that part of the current in phase with the field. As was indicated earlier, two cases are possible: *normal skin effect,* when

$$\nu_0/\omega \gg v/\delta\omega, \quad \text{or} \quad l \ll \delta, \tag{21}$$

and *anomalous skin effect,* when

$$\nu_0/\omega \ll v/\delta\omega, \quad \text{or} \quad l \gg \delta. \tag{22}$$

Prior to calculating the current in the first approximation we recall the inequalities which the frequency must satisfy for our analysis to be cor-

rect. It is assumed here that $\omega_a \sim \Omega$. Then from (1)

$$\nu_0 \ll \omega \ll \Omega,$$

and from $\omega \gg v/\delta$ (the path traversed by the electron during one period of the field is considerably less than the depth of penetration) we find that $\omega \gg \Omega v/c$. Thus

$$\Omega v/c \ll \omega \ll \Omega.$$

According to (21) and (22) we have finally: *normal skin effect*

$$\Omega v/c \ll \nu_0 \ll \omega \ll \Omega;$$

anomalous skin effect

$$\nu_0 \ll \Omega v/c \ll \omega \ll \Omega.$$

Both chains of inequalities can take place, since $v/c \sim 10^{-3}$.

Thus, in case of the normal skin effect we neglect in expression (20) the terms of order $v/\omega\delta$ and retain terms of order ν_0/ω.* For the surface impedance we obtain here

$$X_x = 4\pi\omega\delta_x/c^2; \quad R_x = 2\pi\nu_0\delta_x/c^2. \tag{23}$$

Analogously, if the incident wave is polarized along the y axis:

$$X_y = 4\pi\omega\delta_y/c^2, \quad R_y = 2\pi\nu_0\delta_y/c^2. \tag{23'}$$

To derive (23) and (23') we used the relation between impedance and total current

$$Z_x = E_x(0)/J_x; \quad Z_y = E_y(0)/J_y.$$

For the *anomalous skin effect*, neglecting the terms $\sim \nu_0/\omega$ and retaining the term $\sim v/\delta\omega$, we have from (20):

$$X_x = \frac{4\pi\delta_x\omega}{c^2}; \quad R_x = \frac{3\pi\tilde{v}_x}{4c^2};$$
$$X_y = \frac{4\pi\delta_y\omega}{c^2}; \quad R_y = \frac{3\pi\tilde{v}_y}{4c^2}. \tag{24}$$

Here (the prime stands for integration in the region $v_z > 0$)

$$\tilde{v}_\alpha = \frac{16\oint' \left\{v_z v_\alpha^2 - v_\alpha v_z^2 \oint \frac{v_z v_\alpha}{v}\,dS \Big/ \oint \frac{v_z^2}{v}\,dS\right\}\frac{dS}{v}}{3\left\{\oint v_\alpha^2 \frac{dS}{v} - \left(\oint \frac{v_z v_\alpha}{v}\,dS\right)^2 \Big/ \oint \frac{v_z^2}{v}\,dS\right\}}. \tag{25}$$

In case of a quadratic isotropic dependence $\tilde{v}_\alpha = v$, and we arrive at Eq. (5).

From expression (20) it is evident that even if the incident wave is polarized along one of the main axes of the tensor $(N/m)_{ik}$ the current is elliptically polarized. This, combined with the fact that in the presence of the anomalous skin effect the metal is characterized not by the conductivity tensor but the conductivity operator which obviously cannot be diagonalized at all points.

The presence of elliptical polarization of the current leads to "oblique" terms of surface impedance. Put*

$$E_x(0)/J_{yx} = Z_{xy} \approx R_{xy};$$
$$E_y(0)/J_{xy} = Z_{yx} \approx R_{yx}. \tag{26}$$

Here J_{yx} is the y component of the current, caused by the incidence of an electromagnetic wave polarized along the x axis. Analogously, J_{xy} is the x component of the current due to the electric field J_y.

From (20) and (26):

$$R_{xy} = \frac{\omega^2(2\pi\hbar)^3}{2e^2} \times \left[\oint_{v_z>0} \left(v_y v_x v_z - v_y v_z^2 \oint \frac{v_z v_x}{v}\,dS \Big/ \oint \frac{v_z^2}{v}\,dS\right)\frac{dS}{v}\right]^{-1}, \tag{27}$$

$$R_{yx} = \frac{\omega^2(2\pi\hbar)^3}{2e^2} \times \left[\oint_{v_z>0} \left(v_y v_x v_z - v_x v_z^2 \oint \frac{v_z v_y}{v}\,dS \Big/ \oint \frac{v_z^2}{v}\,dS\right)\frac{dS}{v}\right]^{-1}. \tag{27'}$$

If we assume that the anisotropy is on the order of unity, then from (27) it is easy to obtain an estimate of the "oblique" terms:

$$R_{xy} \sim \omega^2\delta^2/c^2v^2,$$

*It should be taken into account that in this approximation $\delta_{x,y} = \delta^{(0)}_{x,y}(1 - i\nu_0/2\omega)$ where $\delta^{(0)}_{x,y}$ is defined by (16). Equations (23) and (23') contain $\delta^{(0)}_{x,y}$ (zeros omitted!).

*$X_{yx} = X_{xy} = 0$ (zeroth approximation!).

i.e.,

$$R_{xy}/X \sim \delta\omega/v = c\omega/v\Omega \gg 1.$$

Such an inequality is entirely natural since $|J_x| \gg |J_{yx}|$.

Formulas (23)–(25) and (27) solve the postulated problem – determination of expressions for the component of surface impedance in the case of an arbitrary dispersion of conduction electrons, usable in the region of infrared frequences.

3. Consider now the behavior of a wave reflected from a metal. If an electromagnetic wave polarized along the x axis is incident on a metal surface, the electromagnetic field outside the metal has the form

$$\begin{aligned} E_x &= E_{\text{inc}} \cdot e^{-ikz} + E_{\text{ref}}\, e^{ikz}; \\ H_y &= E_{\text{inc}}\, e^{-ikz} - E_{\text{ref}}\, e^{ikz}; \\ E_y &= E_{yx} e^{ikz}; \quad H_x = E_{yx} e^{ikz}; \quad k = \omega/c. \end{aligned} \tag{28}$$

The resultant y component of the electric field and x component of the magnetic field are due to the presence of the "oblique" terms in the surface impedance.

Equations (17) and (26) allow us to calculate the amplitudes of the reflected waves

$$E_{\text{ref}} = -E_{\text{inc}} \frac{1 - cZ_x/4\pi}{1 + cZ_x/4\pi};$$

$$E_{yx} = 2E_{\text{inc}} \frac{Z_x}{Z_{xy}} (1 + cZ_x/4\pi)^{-1};$$

and if we take into account that $|cZ_x/4\pi| \ll 1$, then

$$\begin{aligned} E_{\text{ref}} &\approx -E_{\text{inc}} (1 - cZ_x/2\pi), \\ E_{yx} &\approx 2E_{\text{inc}} Z_x/Z_{xy} \approx 2iE_{\text{inc}} X_x/R_{xy}. \end{aligned} \tag{29}$$

The relationship $E_{yx}/E_{\text{inc}} \sim v\Omega/c\omega$ measures the ellipticity of the reflected wave since Z_x and Z_{xy} have different components (Z_{xy} is real and Z_x almost pure imaginary).

4. Measurement of the surface resistance allows us in principle to obtain information about the form of the Fermi surface. However, the formulas obtained are very complicated, and, naturally, it is not possible to say much about a complete synthesis of the Fermi surface from these data. On the other hand it is possible to use certain simple assumptions about the character of the dispersion dependence, and attempt to make use of the quantity $Z = R + iX$ to determine the constants which describe the properties of the electron gas.

If we examine the isotropic quadratic dependence of energy on momentum*

$$\varepsilon = p^2/2m, \tag{30}$$

then for the measured values of R and X (*cf.* Refs. 3 and 5)† it is possible to determine the effective electron mass m and the electron concentration N:

$$\begin{aligned} N &= 9 \cdot 2^{-3/2} \pi^{5/2} (2\pi\hbar)^{3/2} \left(\frac{\omega}{ec^2}\right)^3 R^{-3/2} X^{-3}; \\ m &= 9 \cdot 2^{-7/2} \pi^{3/2} (2\pi\hbar)^{3/2} (\omega/ec^4) R^{-3/2} X^{-1}. \end{aligned}$$

We use the fact that the limiting velocity of the electrons is determined by the electron concentration and the mass

$$v = (3/8\pi)^{1/3} (2\pi\hbar/m) N^{1/3}.$$

If the static conductivity σ is known, then, in addition to determining N and m, it is possible to determine the collision frequency $\nu_0 = 4\pi\omega^2/c^2X^2\sigma$.

Let us turn attention to the following fact. It is very important to be certain that the formulas used to determine the characteristics of the electron gas [in our case (3) and (5)] are correct; *i.e.*, we must be satisfied that we are really in the plasma region (2) and therefore under the conditions of the anomalous skin effect. For this it is necessary that the ohmic loss be small, *i.e.*, we must have

$$R \ll X. \tag{31}$$

If this condition is fulfilled, we know that we are in the plasma region. The determination of the

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* It is obviously possible to use this dependence for a description of the properties of a polycrystal.

† We, purposely, use only the formulas of the anomalous skin effect since the relaxation time τ (or the collision frequency ν_0) does not enter into them. On the other hand the formulas for the normal skin effect can indeed be used for determining the quantity ν_0 (and, of course, the ratio N/m). Actually from (3) and (4) we have: $\nu_0 = 2\omega R/X$; $N/m = 4\pi\omega^2/e^2c^2X^2$. The value of the static conductivity $\sigma = Ne^2/m\nu_0$ can be employed to check the determined values of N/m and ν_0. Note that in this case it is impossible to determine N and m individually, but only their ratio.

character of the skin effect (normal or anomalous) is difficult since this character is connected with the relationship between the mean free path $l = v\tau$ and the depth of penetration $\delta = c/\Omega$.

If the normal skin effect takes place ($l \ll \delta$) then by using the expression for the surface impedance

$$Z = R + iX = \sqrt{\frac{2\pi\omega}{c^2\sigma}}\{(\sqrt{\omega^2c^2+1}-\omega\tau)^{1/2} + i(\sqrt{\omega^2\tau^2+1}+\omega\tau)^{1/2}\}$$

(*cf.*, for example, Ref. 4; $\sigma = Ne^2\tau/m$ is the static conductivity of the metal) it is easy to find an equation connecting R, X and σ:

$$XRc^2\sigma/2\pi\omega = 1. \tag{32}$$

The relation obtained can serve as a criterion for the normal skin effect since in that case when the mean free path $l = v\tau$ is considerably greater than the depth of penetration of the field, the parameter $\xi = XRc^2\sigma/2\pi\omega$ is also considerably greater than unity. Actually, if we use the expression for surface impedance in the radio-frequency region at low temperatures (the limit of the anomalous skin effect, *cf.*, for example, formula (4.19) of Ref. 1) we obtain

$$\xi \sim (l/\delta)^{2/3} \gg 1 \quad (l \gg c/\sqrt{2\pi\sigma\omega}).$$

From (3) and (5) (infrared region, low temperatures) we have

$$\xi \sim l/\delta \gg 1 \quad (l \gg c/\Omega).$$

Thus, if $\xi \gg 1$ we have the anomalous skin effect. If furthermore $R \ll X$, we deal with the "plasma" region.

Note that allowing for the polarization of the bound electrons (electrons of the ionic residues of the metal) does not change substantially the value of the parameter ξ. For instance, under the conditions of the normal skin effect we have for $\omega \ll \Omega$:

$$\xi = 1 + 2\varepsilon_0\omega^2/\Omega^2 \approx 1.$$

5. Generally in optical investigations we do not make use of the surface impedance, but use the idea of a complex coefficient of reflection $n - i\varkappa$. When the mean free path of the electron is much less than the depth of the skin layer ($l \ll \delta$, normal skin effect) the coefficient of reflection is connected in the usual manner with the complex dielectric constant

$$n - i\varkappa = \sqrt{\varepsilon'}; \quad \varepsilon' = \varepsilon(\omega) - \pi 4\sigma(\omega)\, i/\omega.$$

In particular, if the relaxation effect is described with the help of the relaxation time ($\sigma = \sigma_0/(1+i\omega\tau)$) and we do not take into account the ionic residues, then

$$\varepsilon(\omega) = 1 - \Omega^2/(\nu_0^2+\omega^2); \quad \sigma(\omega) = \sigma_0\nu_0^2/(\nu_0^2+\omega^2);$$

$$n = \left\{\frac{1}{2}\left(1 - \frac{\Omega^2}{\nu_0^2+\omega^2}\right) + \frac{1}{2}\sqrt{\left(1-\frac{\Omega^2}{\nu_0^2+\omega^2}\right)^2 + \left(\frac{\nu_0}{\omega}\frac{\Omega^2}{\nu_0^2+\omega^2}\right)^2}\right\}^{1/2};$$

$$\varkappa = \left\{\frac{1}{2}\left(\frac{\Omega^2}{\nu_0^2+\omega^2} - 1\right) + \frac{1}{2}\sqrt{\left(1-\frac{\Omega^2}{\nu_0^2+\omega^2}\right)^2 + \left(\frac{\nu_0}{\omega}\frac{\Omega^2}{\nu_0^2+\omega^2}\right)^2}\right\}^{1/2}.$$

The expression for the surface impedance, as is known,[1,9] may serve as a basis for introducing the effective dielectric constant ε_{eff}, and consequently n_{eff} and $\varkappa_{\text{eff}}$;*

$$n_{\text{eff}} = \frac{cR/4\pi}{(cR/4\pi)^2 + (cX/4\pi)^2}, \qquad \varkappa_{\text{eff}} = \frac{cX/4\pi}{(cR/4\pi)^2 + (cX/4\pi)^2}. \tag{33}$$

For $R \ll X$ (of interest in our case)

$$n_{\text{eff}} = 4\pi R/cX^2, \quad \varkappa_{\text{eff}} = 4\pi/cX. \tag{33'}$$

For expression (33) to be considered as defining the effective coefficient of reflection, the impedance must depend weakly on the form of the incident wave, *i.e.*, on the angle of incidence and polarization.[1,9] In order to be satisfied that the concept of effective dielectric constant can be used also in the infrared region (under the conditions of the anomalous skin effect) we have examined the dependence of the surface impedance on the angle of incidence φ in that case when the vector of the electric field is polarized in the plane of incidence (p is the polarization). Assuming the quadratic dispersion law (30), we have obtained ($\omega/\Omega \ll 1$):

* For $\delta \gg l$, naturally, $n_{\text{eff}} = n$, $\varkappa_{\text{eff}} = \varkappa$.

$$X = \frac{4\pi}{c}\frac{\omega}{\Omega}\left(1 + \frac{v^2}{5c^2}\sin^2\varphi\right), \quad R = \frac{3\pi}{4}\frac{v}{c^2} \qquad (34)$$

Thus, in this case it is not necessary to take into account the dependence on the angle of incidence, since $v^2/c^2 \sim 10^{-6}$, and we have discarded terms known to be larger to obtain formulas (3), (5), (23), (24), and (34).

Therefore, in the region of frequency and temperature under consideration we may introduce an effective index of reflection, which

$$n_{\text{eff}} = 3v\Omega^2/16c\omega^2; \quad \varkappa_{\text{eff}} = \Omega/\omega \qquad (35)$$

[we have used (33′)]. Note that in this region one must satisfy a condition analogous to (31):

$$n_{\text{eff}} \ll \varkappa_{\text{eff}}, \qquad (31')$$

In the anisotropic case (particularly, when the ohmic resistance has "oblique" terms) introduction of an effective index of reflection is inconvenient and does not present great interest.

6. In the case of a single-crystal metal with anisotropic conductivity we use for the description of the electron gas a quadratic dispersion law with two effective masses:

$$\varepsilon = (p_1^2 + p_2^2)/2m_1 + p_3^2/2m_3. \qquad (36)$$

Assuming that the normal to the surface makes an angle α with the axis of the crystal, we obtain:

(1) The coordinate of the principal directions* on the surface of the metal

$$\mathbf{x} = [\mathbf{z}\times[\mathbf{z}\times\mathbf{n}]]/\sin\alpha; \quad \mathbf{y} = [\mathbf{z}\times\mathbf{n}]/\sin\alpha$$

($\mathbf{z}$ is the coordinate of the normal to the surface, $\mathbf{n}$ the coordinate of the axis of the crystal, and $\cos\alpha = (\mathbf{z}\cdot\mathbf{n})$.

(2) Principal values of the tensor $(N/m)_{ik}$:

$$\left(\frac{N}{m}\right)_{xx} = \frac{N}{m_1\cos^2\alpha + m_3\sin^2\alpha}; \quad \left(\frac{N}{m}\right)_{yy} = \frac{N}{m_1}$$

(3) Components of the surface impedance:

$$X_{xx} = \left[\frac{4\pi\omega^2}{Ne^2c^2}(m_1\cos^2\alpha + m_3\sin^2\alpha)\right]^{1/2},$$
$$X_{yy} = \sqrt{\frac{4\pi\omega^2}{Ne^2c^2}m_1},$$
$$R_{xx} = R_{yy} = \frac{3\pi}{4c^2}\left[2\varepsilon_0\left(\frac{\cos^2\alpha}{m_3} + \frac{\sin^2\alpha}{m_1}\right)\right]^{1/2}; \qquad (37)$$
$$\varepsilon_0 = \left(\frac{3N}{8\pi}\right)^{2/3}\frac{(2\pi\hbar)^2}{2m_1^{2/3}m_3^{1/3}}$$

(ε_0 is the limiting Fermi energy). The oblique terms of the surface impedance R_{xy} and R_{yx} in this case are equal to zero. This, obviously, is connected with the "isotropy" of R_{ik}. It should be borne in mind that the vanishing of the "oblique" terms of the ohmic resistance in this simple case does not at all indicate that they are equal to zero in reality. The point is that the selected dispersion law (36), although it also may describe, for example, the anisotropy of the conductivity, is known not to be useful for more complex effects (for example, galvanomagnetism, de Hass–van Alphen effect, *etc.*). Obviously, for the description of the formation of electric polarization at normal incidence a more complex dispersion law is necessary. For example, the assumption of some specially oriented ellipsoid (as in Bi)[10] leads to the expected effect.

Equations (37) allow us to determine the electron concentration N and both values of the effective mass (m_1, m_3):

$$N = \frac{9\pi^{5/2}}{2\sqrt{2}}(2\pi\hbar)^{3/2}\left(\frac{\omega}{ec^2}\right)^3\frac{1}{X_y^{5/2}X_x^{1/2}R^{3/2}}\frac{\sin^2\alpha}{1-(X_y/X_x)^2\cos^2\alpha};$$
$$m_1 = \frac{9\pi^{3/2}}{8\sqrt{2}}(2\pi\hbar)^{3/2}\frac{\omega}{ec^4}\frac{1}{X_y^{1/2}X_x^{1/2}R^{3/2}}\frac{\sin^2\alpha}{1-(X_y/X_x)^2\cos^2\alpha};$$
$$m_3 = (9\pi^{3/2}/8\sqrt{2})(2\pi\hbar)^{3/2}(\omega/ec^4)X_x^{3/2}/X_y^{5/2}R^{3/2}.$$

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*I.e., the direction in which the tensor X_{ik} is diagonal (*cf.* Sec. 2).

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Bremsstrahlung of Ultra-Relativistic Particles in a Central Field

A. G. SITENKO
Khar'kov State University

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Bremsstrahlung of an ultra-relativistic particle of spin one-half in an arbitrary field with central symmetry is considered. A relation between the bremsstrahlung cross section and elastic scattering cross section is obtained for ultra-relativistic particles.

1. AS IS KNOWN, the bremsstrahlung in the collision of an ultra-relativistic charged particle with a nucleus occurs principally at large distances from the nucleus. The cross section for the process is therefore determined by the assymptotic form of the wave function of the particle in the nuclear field.[1-3] The assymptotic form of the particle wave functions may be found by describing the scattering effect of the nucleus by the scattering matrix. In the ultra-relativistic case, it is possible to establish a general relationship between bremsstrahlung and elastic scattering cross sections. This relationship does not depend on the character of the interaction between the particles and the scattering nuclear field.

Let us first of all consider the elastic scattering of fast particles with spin ½ in a field with central symmetry. The free motion of a spin-½ particle of momentum p is described by the spinor plane wave

$$\psi_0 = u_p e^{i\mathbf{pr}},$$

where u_p is the unit amplitude of the spinor wave. The scattering of particles in an external central field will be characterized by the scattering matrix S. The wave function describing the stationary states of the particles in the external field will then obviously be determined by the product of the matrix S by ψ_0

$$\psi_{\mathbf{p}}^{(+)} = S\psi_0. \tag{1}$$

At large distances from the center of the field, the wave function (1) will be of the form of a sum of a plane wave and an outgoing spherical wave. To verify this, let us use the Huygens principle as formulated by Akhiezer[4] for spinor waves. This principle establishes a relationship between the value of the wave function at a certain point and the value of the wave function on a closed surface surrounding this point. Let us choose for this surface an infinite plane perpendicular to the momentum of the impinging particle and passing through the center of the external field

$$\begin{aligned}\psi_{\mathbf{p}}^{(+)}(\mathbf{r}) &= \frac{1}{4\pi}\int\left(\gamma\frac{\partial}{\partial\mathbf{r}} - \gamma_4 E - m\right)\gamma\mathbf{n}\frac{\exp(ip|\mathbf{r}-\boldsymbol{\rho}|)}{|\mathbf{r}-\boldsymbol{\rho}|}Su_p e^{i\mathbf{p}\boldsymbol{\rho}}\,d\boldsymbol{\rho}\\ &= u_p e^{i\mathbf{pr}} - \frac{1}{4\pi}\int\left(\gamma\frac{\partial}{\partial\mathbf{r}} - \gamma_4 E - m\right)\gamma\mathbf{n}\frac{\exp(ip|\mathbf{r}-\boldsymbol{\rho}|)}{|\mathbf{r}-\boldsymbol{\rho}|}\{1-S\}u_p e^{i\mathbf{p}\boldsymbol{\rho}}\,d\boldsymbol{\rho}.\end{aligned} \tag{2}$$

Far from the center of the field ($r \to \infty$), this function has the form