value. The probability of transition into higher states depends appreciably on the boundary condition.

2. MINIMUM ANGULAR WIDTH OF THE WAVE FUNCTION

The action of centrifugal forces and the energy loss of the α-particle (nuclear rotation) lead to a finite width of the wave function on the sphere of exit from underneath the barrier, even for a δ-function distribution of α-particles on the sphere Σ. The intensities of the fine-structure lines in the α-spectrum for a deformed nucleus cannot, therefore, exceed the values given by the simple Gamow formula, which does not take into account the nuclear deformation. The minimum width of the wave function can be obtained either by using formula (A.5), putting $c = \infty$, or directly from Eq. (10), by letting $v_1 = 0$ and $y_1(x_0) = \infty$ in the Ricatti equation. We obtain

$$y_1(x^*)\big|_{\max} = \Big[2\int\limits_{x_0}^{x^*} \frac{(\beta + x^{-2})}{(v_0 - 1)^{1/2}}\Big]^{-1} = \frac{1}{2\,dk_0R_0}. \quad (A.6)$$

The angular half width ϑ^* of the wave function is determined from $\sin\vartheta^* \approx \sqrt{k_0R_0y_1(x^*)}$; for the values of d mentioned above and for $k_0R_0 \approx 8$, it is equal to $\sim 40°$. As can be seen from Eq. (10), the quadrupole potential can only decrease the value of $y_1(x^*)$ and leads, therefore, to an additional increase of the angular width of the wave function and to a decrease of the probability of transition into excited rotational states.

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Motion of a Charge Parallel to the Axis of a Cylindrical Channel in a Dieletric

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The field produced by a charge moving parallel to the axis of a cylindrical channel in a dielectric is determined. The field and energy losses of the charge are computed for various assumptions concerning the medium.

THE PASSAGE OF A CHARGE along a channel in a dielectric was first considered by Ginzburg and Frank.[1] These authors calculated the field produced by a point charge moving with uniform motion along the axis of a cylindrical channel of radius a, filled with a dielectric $\varepsilon_1(\omega)$ in a medium of dielectric constant $\varepsilon_2(\omega)$.

Problems connected with the passage of a charge along the axis of a channel in a dielectric have also been treated by Bohr,[2] Schoenberg,[3] Huybrechts,[3,5] and Sitenko[6] (problems of this type have also been considered in Ref. 7).

In problems concerning the generation of electromagnetic radiation, focusing of charged particles in a cylindrical channel, and the theory of Cerenkov counters, it is of interest to consider the case in which the charge moves in the channel along a straight line parallel to the axis rather than along the axis itself.

1. We consider an infinite cylindrical channel in an isotropic medium of dielectric constant $\varepsilon_2(\omega)$. It will be assumed that the space inside the channel is filled with an isotropic dielectric with dielectric constant $\varepsilon_1(\omega)$. The radius of the channel is denoted by a. We consider a charge which moves with uniform motion along a line parallel to the axis of a channel at a distance r_0. The equations for the electromagnetic field potentials are as follows:

$$\Delta\varphi = -\frac{4\pi\rho}{\varepsilon},$$
$$\Delta\mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} + \frac{\varepsilon}{c}\operatorname{grad}\frac{\partial\varphi}{\partial t}. \quad (1)$$

In what follows we shall choose the potential gauge to make $\operatorname{div}\mathbf{A} = 0$. In Eq. (1) $\varepsilon = \varepsilon_1$ inside the channel and $\varepsilon = \varepsilon_2$ outside. We introduce the cylindrical coordinates r, φ, and z with the z axis parallel to the channel axis. The angle φ is measured from the plane which passes through the axis of the channel and the line of motion of the charge. Then, in Eq. (1) the charge density ρ is given by:

$$\rho = e\delta(\mathbf{r} - \mathbf{r}_0)\,\delta(z - vt). \quad (2)$$

Here $\mathbf{r}_0$ is the projection of the radius vector of the particle on the plane z = const, while $\mathbf{r}$ is the projection of the radius vector of the point of observation on the same plane. In the present problem it is convenient to represent $\delta(\mathbf{r} - \mathbf{r}_0)$ in the following form:

$$\delta(\mathbf{r} - \mathbf{r}_0) = \frac{1}{2\pi}\sum_{m=-\infty}^{m=\infty} e^{im\varphi}\int_0^\infty J_m(kr)\,J_m(kr_0)\,k\,dk, \quad (3)$$

where $J_m(x)$ is a Bessel function of order m and φ is the angle between $\mathbf{r}$ and $\mathbf{r}_0$. For $\delta(z - vt)$ we use the usual representation

$$\delta(z - vt) = \frac{1}{2\pi v}\int e^{i\omega(z-vt)/v}\,d\omega. \quad (4)$$

Taking account of (3) and (4), Eq. (2) can be rewritten in the form

$$\rho = \frac{e}{(2\pi)^2 v}\sum_{m=-\infty}^{m=\infty} e^{im\varphi} \times \int J_m(kr)\,J_m(kr_0)\,e^{i\omega(z-vt)/v}k\,dk\,d\omega. \quad (5)$$

Hereinafter, when the limits of integration are not given explicitly, the integration over k is taken from 0 to ∞ while the integration over ω is taken from $-\infty$ to $+\infty$.

We find the solution for the first equation in (1). The expression for ρ in (5) is an expansion in the eigenfunctions of the Laplacian operator in cylindrical coordinates. This facilitates the determination of the particular solution φ_{inhom} of the inhomogeneous equation. If we write ρ in the form

$$\rho = \frac{e}{(2\pi)^2 v}\sum_{m=-\infty}^{m=\infty}\int \rho_m(r, z, \varphi, t, k, \omega)\,k\,dk\,d\omega, \quad (6)$$

then from Eq. (5), ρ_m satisfies the equation $\Delta\rho_m = \lambda\rho_m$, *i.e.*, it is an eigenfunction of the Laplacian operator with eigenvalues $\lambda = -(k^2 + \omega^2/v^2)$. Whence it follows that

$$\varphi = \frac{e}{\pi v}\sum_{m=-\infty}^{m=\infty} e^{im\varphi}\int\frac{J_m(kr)\,J_m(kr_0)}{\varepsilon(\omega)\,(k^2+\omega^2/v^2)}\,e^{i\omega(z-vt)/v}k\,dk\,d\omega. \quad (7)$$

The integration over k in Eq. (7) is performed without difficulty. We obtain

$$\varphi = \frac{e}{\pi v}\sum_{m=-\infty}^{m=\infty} e^{im\varphi}\int e^{i\omega(z-vt)/v}\varphi_{m\,\text{inhom}}(r, \omega)\,d\omega, \quad (8)$$

$$\varphi_{m\,\text{inhom}} = \begin{cases} \dfrac{1}{\varepsilon_1(\omega)} K_m\left(\dfrac{|\omega|}{v}r\right) I_m\left(\dfrac{|\omega|}{v}r_0\right) & r > r_0 \\ \dfrac{1}{\varepsilon_1(\omega)} I_m\left(\dfrac{|\omega|}{v}r\right) K_m\left(\dfrac{|\omega|}{v}r_0\right) & r < r_0. \end{cases} \quad (9)$$

Here, I_m and K_m are Bessel functions of imaginary argument. Eq. (8) gives the particular solution of the inhomogeneous equation inside the channel. To this solution it is necessary to add a solution of the homogeneous equation which satisfies the boundary conditions at $r = a$.

To the solution in (9) we add the function $\varphi_{m\,\text{hom}}$ which satisfies

$$\varphi_{m\,\text{hom}} = \begin{cases} \dfrac{\alpha_m}{\varepsilon_1} I_m\left(\dfrac{|\omega|}{v}r_0\right) I_m\left(\dfrac{|\omega|}{v}r\right) & r < a \\ \gamma_m K_m\left(\dfrac{|\omega|}{v}r\right) & r > a. \end{cases} \quad (10)$$

The boundary conditions at $r = a$

$$\varphi_{1m} = \varphi_{2m}, \quad \varepsilon_1\partial\varphi_{1m}/\partial r = \varepsilon_2\partial\varphi_{2m}/\partial r$$

yield

$$\alpha_m = \frac{(\varepsilon_2 - \varepsilon_1) K_m(|\omega| a/v) K'_m(|\omega| a/v)}{\varepsilon_1 K_m(|\omega| a/v) I'_m(|\omega| a/v) - \varepsilon_2 I_m(|\omega| a/v) K'_m(|\omega| a/v)};$$
$$\gamma_m = \frac{(v/\omega a) I_m(|\omega| r_0/v)}{\varepsilon_1 K_m(|\omega| a/v) I'_m(|\omega| a/v) - \varepsilon_2 I_m(|\omega| a/v) K'_m(|\omega| a/v)}. \tag{11}$$

Thus the potential of the longitudinal field is completely determined. The expansion of the potential is of the form given in (8) where instead of $\varphi_{m\,\text{inhom}}$ we must use $\varphi_{m\,\text{inhom}} + \varphi_{m\,\text{hom}}$.

We now ascertain the vector potential $\mathbf{A}$ for the transverse field. First we find the particular solution for the inhomogenous equation

$$\Delta \mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} + \frac{\varepsilon}{c}\operatorname{grad}\frac{\partial \varphi}{\partial t}.$$

The solution of this equation is conveniently represented in the form

$$\mathbf{A}_{\text{inhom}} = \mathbf{A}_{\text{I}} + \mathbf{A}_{\text{II}}, \tag{12}$$

$$\mathbf{A}_{\text{I}} = \frac{e}{\pi c}\sum_{m=-\infty}^{m=+\infty} e^{im\varphi}\int e^{i\omega(z-vt)/v}\,\mathbf{A}_{m\text{I}}(r, \omega)\, d\omega,$$

$$A_{mz\text{I}} = \frac{1}{\varepsilon_1\beta^2}\left[K_m\left(\frac{|\omega|}{v}r\right) I_m\left(\frac{|\omega|}{v}r_0\right) - \varkappa_1^2 K_m\left(\frac{|\omega|}{v}\varkappa_1 r\right) I_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)\right],$$
$$A_{mr\text{I}} = -\frac{i}{\varepsilon_1\beta^2}\left[K'_m\left(\frac{|\omega|}{v}r\right) I_m\left(\frac{|\omega|}{v}r_0\right) - \varkappa_1 K'_m\left(\frac{|\omega|}{v}\varkappa_1 r\right) I_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)\right], \tag{13}$$
$$A_{m\varphi\text{I}} = \frac{mv}{\omega r}\frac{1}{\varepsilon_1\beta^2}\left[K_m\left(\frac{|\omega|}{v}r\right) I_m\left(\frac{|\omega|}{v}r_0\right) - K_m\left(\frac{|\omega|}{v}\varkappa_1 r\right) I_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)\right],$$
$$\varkappa_1 = \sqrt{1-\varepsilon_1\beta^2},\quad \omega > 0,\quad r > r_0,\ r < a.$$

Equation (13) applies for $\omega > 0$, $r > r_0$. For $r < r_0$, the expressions for $\mathbf{A}_m$ are obtained from (13) by replacing I_m with K_m, K'_m with I'_m, and K_m with I_m. For example, for $r < r_0$, $\omega > 0$:

$$A_{mr\text{I}} = -\frac{i}{\varepsilon_1\beta^2}\left[I'_m\left(\frac{|\omega|}{v}r\right) K_m\left(\frac{|\omega|}{v}r_0\right) - \varkappa_1 I'_m\left(\frac{|\omega|}{v}\varkappa_1 r\right) K_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)\right].$$

When $\omega < 0$, the complex conjugate of the expression is used. In this case, if $\varepsilon_1\beta^2 > 1$,

$$\varkappa_1 = \begin{cases} +i\sqrt{\varepsilon_1\beta^2 - 1} & \omega > 0 \\ -i\sqrt{\varepsilon_1\beta^2 - 1} & \omega < 0 \end{cases}$$

The expressions in (13) also apply when $r > a$; in this case ε_1 is replaced by ε_2.

The term $\mathbf{A}_{\text{II}}$ is determined by the expression

$$\mathbf{A}_{\text{II}} = \frac{e}{\pi c}\sum_{m=-\infty}^{m=+\infty} e^{im\varphi}\int e^{i\omega(z-vt)/v}\,\mathbf{A}_{m\text{II}}(r, \omega)\, d\omega,$$

where

$$A_{mz\text{II}} = \begin{cases} \dfrac{\alpha_m}{\varepsilon_1\beta^2} I_m\left(\dfrac{|\omega|}{v}r_0\right) I_m\left(\dfrac{|\omega|}{v}r\right) & r < a, \\ \dfrac{\gamma_m}{\beta^2} K_m\left(\dfrac{|\omega|}{v}r\right) & r > a, \end{cases} \qquad A_{mr\text{II}} = \begin{cases} -i\dfrac{\alpha_m}{\varepsilon_1\beta^2} I_m\left(\dfrac{|\omega|}{v}r_0\right) I'_m\left(\dfrac{|\omega|}{v}r\right) & r < a \\ -i\dfrac{\gamma_m}{\beta^2} K'_m\left(\dfrac{|\omega|}{v}r\right) & r > a, \end{cases}$$

$$A_{m\varphi\text{II}} = \begin{cases} \dfrac{mv}{\omega r}\dfrac{\alpha_m}{\varepsilon_1\beta^2} I_m\left(\dfrac{|\omega|}{v}r_0\right) I_m\left(\dfrac{|\omega|}{v}r\right) & r < a \\ \dfrac{mv}{\omega r}\dfrac{\gamma_m}{\beta^2} K_m\left(\dfrac{|\omega|}{v}r\right) & r > a. \end{cases} \tag{14}$$

Equations (13) and (14) yield the particular solution of the inhomogeneous equation inside and outside the channel.

We now find the general solution for the homogeneous equation. It is necessary to find the general expression for the "cylindrical" vector, *i.e.*, the vector which satisfies the following equation in a cylindrical coordinate system:

$$\mathbf{A}_{\text{hom}} = 0, \quad \operatorname{div} \mathbf{A}_{\text{hom}} = 0.$$

It is easily shown that this vector can be written in the form

$$\mathbf{A}_{\text{hom}} = \frac{e}{\pi c} \sum_{m=-\infty}^{m=\infty} e^{im\varphi} \int e^{i\omega(z-vt)/v} \mathbf{A}_{m\,\text{hom}}(r, \omega)\, d\omega,$$

$$A_{mz\,\text{hom}} = \begin{cases} \dfrac{-\varkappa_1^2}{\varepsilon_1\beta^2} \lambda_{1m} I_m\left(\dfrac{|\omega|}{v}\varkappa_1 r_0\right) I_m\left(\dfrac{|\omega|}{v}\varkappa_1 r\right) & r < a \\[2ex] -\dfrac{\varkappa_2^2}{\varepsilon_2\beta^2} \lambda_{2m} I_m\left(\dfrac{|\omega|}{v}\varkappa_2 r_0\right) K_m\left(\dfrac{|\omega|}{v}\varkappa_2 r\right) & r > a, \end{cases}$$

$$A_{mr\,\text{hom}} = \begin{cases} \dfrac{i\varkappa_1}{\varepsilon_1\beta^2} I_m\left(\dfrac{|\omega|}{v}\varkappa_1 r_0\right)\left[\lambda_{1m} I'_m\left(\dfrac{|\omega|}{v}\varkappa_1 r\right) + \dfrac{mv}{\omega r}\dfrac{\theta_{1m}}{\varkappa_1} I_m\left(\dfrac{|\omega|}{v}\varkappa_1 r\right)\right] & r < a \\[2ex] \dfrac{i\varkappa_2}{\varepsilon_2\beta^2} I_m\left(\dfrac{|\omega|}{v}\varkappa_2 r_0\right)\left[\lambda_{2m} K'_m\left(\dfrac{|\omega|}{v}\varkappa_2 r\right) + \dfrac{mv}{\omega r}\dfrac{\theta_{2m}}{\varkappa_2} K_m\left(\dfrac{|\omega|}{v}\varkappa_2 r\right)\right] & r > a \end{cases} \tag{15}$$

$$A_{m\varphi\,\text{hom}} = \begin{cases} -\dfrac{\varkappa_1}{\varepsilon_1\beta^2} I_m\left(\dfrac{|\omega|}{v}\varkappa_1 r_0\right)\left[\dfrac{mv}{\omega r}\dfrac{\lambda_{1m}}{\varkappa_1} I_m\left(\dfrac{|\omega|}{v}\varkappa_1 r\right) + \theta_{1m} I'_m\left(\dfrac{|\omega|}{v}\varkappa_1 r\right)\right] & r < a \\[2ex] \dfrac{-\varkappa_2}{\varepsilon_1\beta^2} I_m\left(\dfrac{|\omega|}{v}\varkappa_2 r_0\right)\left[\dfrac{mv}{\omega r}\dfrac{\lambda_{2m}}{\varkappa_2} K_m\left(\dfrac{|\omega|}{v}\varkappa_2 r\right) + \theta_{2m} K'_m\left(\dfrac{|\omega|}{v}\varkappa_2 r\right)\right] & r > a. \end{cases}$$

The coefficients λ_{1m}, λ_{2m}, θ_{1m} and θ_{2m} must be determined from the continuity conditions on E_z, E_φ, E_r and H_z at the edge of the channel. These conditions yield four equations for the coefficients. For brevity we shall use the following notation

$$I_m\left(\frac{|\omega|}{v} a\right) = I, \quad I_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right) = I_{\varkappa_1}, \quad K_m\left(\frac{|\omega|}{v}\varkappa_1 a\right) = K_1,$$
$$K_m\left(\frac{|\omega|}{v} a\right) = K, \quad I_m\left(\frac{|\omega|}{v}\varkappa_2 r_0\right) = I_{\varkappa_2}, \quad K_m\left(\frac{|\omega|}{v}\varkappa_2 a\right) = K_2,$$
$$I_m\left(\frac{|\omega|}{v}\varkappa_1 a\right) = I_1.$$

Then,

$$\lambda_{1m} = \frac{1}{D}\left[(\varkappa_1\varkappa_2)^2(\varkappa_1 I_1 K'_2 - \varkappa_2 I'_1 K_2)(\varkappa_1\varepsilon_2 K_1 K'_2 - \varkappa_2\varepsilon_1 K_2 K'_1) - \left(\frac{mv}{\omega a}\right)^2 I_1 K_1 K_2^2 (\varepsilon_2 - \varepsilon_1)^2 \beta^2\right],$$

$$\lambda_{2m} = \frac{1}{D}\left[\varkappa_1^3\varkappa_2\varepsilon_2 \frac{v}{\omega a}\frac{I_{\varkappa_1}}{I_{\varkappa_2}}(\varkappa_1 I_1 K'_2 - \varkappa_2 I'_1 K_2)\right] - 1;$$

$$\theta_{1m} = \frac{1}{D}\left[m\left(\frac{v}{\omega a}\right)^2 \varkappa_2^2\varepsilon_1(\varepsilon_1 - \varepsilon_2)\beta^2 K_2^2\right];$$

$$\theta_{2m} = \frac{1}{D}\left[m\left(\frac{v}{\omega a}\right)^2 \varkappa_1^2\varepsilon_2(\varepsilon_1 - \varepsilon_2)\beta^2 I_1 K_2 \frac{I_{\varkappa_1}}{I_{\varkappa_2}}\right];$$

$$D = (\varkappa_1\varkappa_2)^2(\varkappa_1 I_1 K'_2 - \varkappa_2 I'_1 K_2)(\varepsilon_1\varkappa_2 K_2 I'_1 - \varepsilon_2\varkappa_1 I_1 K'_2) + \left(\frac{mv}{\omega a}\right)^2 (I_1 K_2)^2 (\varepsilon_2 - \varepsilon_1)^2 \beta^2. \tag{16}$$

Equations (13), (14), (15) and (16) determine completely the vector potential of the transverse field which satisfies the boundary conditions. If, in we set $r_0 = 0$ the formulas obtained earlier for φ and $\mathbf{A}$ these equations pertain to the motion of a charge along the axis of a channel of radius a. This procedure is easily carried out, noting that

$$I_m(0) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0, \end{cases}$$

while $mI_m(0) = 0$ for any m. As is to be expected, the formulas for φ and $\mathbf{A}$ go over to expressions for the potentials obtained in Refs. 1 and 7 in which the motion of a charge along the axis of a channel in a dielectric was considered.

2. Using the gauge which is obtained from the condition $\operatorname{div}\mathbf{A} = 0$, it is possible to divide the field into a longitudinal part and a transverse part. For example,

$$E_{z\,\text{long}} = -\frac{\partial\varphi}{\partial z}, \quad E_{z\,\text{tr}} = -\frac{1}{c}\frac{\partial A_z}{\partial t} \tag{17}$$

In what follows, it will be of interest to find the energy lost by the charge in the excitation of the transverse field in the medium, in particular, the loss due to Cerenkov radiation. This energy loss can be determined in terms of the reaction on the charge itself of the transverse field created by the charge

$$eE_{z\text{tr}} = \frac{e^2}{\pi v^2}\sum_{m=0}^{\infty} a_m \int \frac{i\omega}{\varepsilon_1}\Big\{ I_m\Big(\frac{|\omega|}{v} r_0\Big)\Big[K_m\Big(\frac{|\omega|}{v} r_0\Big) + \alpha_m I_m\Big(\frac{|\omega|}{v} r_0\Big)\Big] - \varkappa_1^2 I_m\Big(\frac{|\omega|}{v}\varkappa_1 r_0\Big)\Big[K_m\Big(\frac{|\omega|}{v}\varkappa_1 r_0\Big) + \lambda_{1m} I_m\Big(\frac{|\omega|}{v}\varkappa_1 r_0\Big)\Big]\Big\} d\omega, \tag{18}$$

where $a_m = 1$ for $m = 0$ and $a_m = 2$ for $m \neq 0$.

For values of ω which satisfy $\varepsilon_1(\omega) = 0$, the expression under the integral sign has no singularity. This is to be expected since the losses at the frequencies for which $\varepsilon_1(\omega) = 0$ are characteristic only of the longitudinal field.

To compute the dependence of the energy integral on r_0, we consider the case analyzed by Ginzburg and Frank,[1] *i.e.*, the motion of a charge inside a hollow channel when

$$\varepsilon_1 = 1, \ \varepsilon_2\beta^2 > 1.$$

Since $\varkappa_1 = \sqrt{1-\beta^2} > 0$, I_m and K_m are real functions; hence the energy loss (18) for the present case can be expressed as follows:

$$eE_{z\text{tr}} = -\frac{e^2}{\pi v^2}(1-\beta^2)\sum_{m=0}^{\infty} a_m \times \int_{\varepsilon_2\beta^2>1} \lambda_{1m} I_m^2\Big(\frac{|\omega|}{v}\varkappa_1 r_0\Big) i\omega\, d\omega,$$

where λ_{1m} is determined from (16).

If the departure of the charge from the axis of the channel is small, the energy integral is simplified if I_m is expanded in powers of r_0

$$eE_{z\text{tr}} = -\frac{e^2}{\pi v^2}(1-\beta^2)\int_{\varepsilon_2\beta^2>1}\lambda_{10} i\omega\, d\omega - \frac{e^2}{\pi v^2}(1-\beta^2) r_0^2 \int_{\varepsilon_2\beta^2>1}\frac{\lambda_{10}+\lambda_{11}}{2} i\omega^3 d\omega. \tag{19}$$

Using well-known relations for Bessel functions, it may be shown that the first term in (19) coincides with the expression obtained in Ref. 1. The second term in (19) is proportional to r_0^2. This dependence on r_0 at small r_0 indicates that when the charge moves along the axis of the channel the loss is either a maximum or a minimum. The coefficient λ_{11} [*cf.* (16)] is a complicated complex function and, in the general case, an investigation of the expression in (19) is difficult.

3. We consider the motion of a charge in the particular case in which the second medium is a metal with infinite conductivity. This case is characterized by $\varkappa_2 \to \infty$ and $\varepsilon_2 \to -\infty$.

It is easy to show that in the case at hand (18) goes over to the expression

$$eE_{z\,\text{tr}} = -\frac{e^2}{2v^2}\operatorname{Re}\sum_{m=0}^{\infty} a_m \int_{\varepsilon_1\beta^2>1}\frac{1-\varepsilon_1\beta^2}{\varepsilon_1}\frac{J_m^2(|\omega||\varkappa_1|r_0/v)\,N_m(|\omega||\varkappa_1|a/v)}{J_m(|\omega||\varkappa_1|a/v)} i\omega\, d\omega. \tag{20}$$

The loss integral is determined by the residues at the poles of the expression under the integral. Because of the poles the integrals become series and the continuous spectrum is replaced by a discrete spectrum characteristic of a waveguide.

Actually a pole is found when $J_m\Big(\frac{|\omega|}{v}\sqrt{\varepsilon_1\beta^2 - 1}\,a\Big) = 0$, *i.e.*, for values $\omega = \omega_{ms}$, for which $\frac{|\omega_{ms}|}{v}\sqrt{\varepsilon_1(\omega_{ms})\beta^2 - 1}\,a = \mu_{ms}$ where μ_{ms} is the sth root of the Bessel function of mth order. The following expression is obtained for the loss due to Cerenkov radiation:

$$eE_{z\text{tr}} = -\frac{2e^2}{a^2}\sum_{m,s} a_m \frac{\mu_{ms}}{\omega_{ms}} \tag{21}$$

$$\times\left[\frac{d}{d\omega}\left(\frac{|\omega|}{v}\sqrt{\varepsilon_1\beta^2-1}\,a\right)\right]^{-1}_{\omega=\omega_{ms}} \frac{1}{\varepsilon_1(\omega_{ms})}\frac{J_m^2(\mu_{ms}r_0/a)}{J_m'^2(\mu_{ms})}.$$

If the material with which the wave guide is filled is dispersionless, that is, if $\varepsilon_1 = \text{const} > 1/\beta^2$, the loss can be written in a simpler form

$$eE_z = -\frac{4e^2}{a^2\varepsilon_1}\sum_{s=1}^{\infty}\left[\frac{1}{2}\frac{J_0^2(\mu_{0s}r_0/a)}{J_0'^2(\mu_{0s})} + \sum_{m=1}^{\infty}\frac{J_m^2(\mu_{ms}r_0/a)}{J_m'^2(\mu_{ms})}\right]. \tag{22}$$

Eq. (22) was obtained by a different method in Ref. 8. If the charge moves along the axis the formula for the loss due to Cerenkov radiation goes over to the expression first obtained by Akhiezer, Liubarskii, and Fainberg:

$$eE_{z\text{tr}} = -\frac{2e^2}{a^2}\frac{1}{\varepsilon_1}\sum_{s=1}^{\infty} J_1^{-2}(\mu_{0s}) \quad \text{for} \quad \varepsilon_1\beta^2 > 1.$$

Using Eq. (22) we can determine the magnitude of the Cerenkov-radiation loss as a function of r_0. The Cerenkov loss is a maximum when $r_0 = 0$. Actually, it is easy to show that in this case, when $r_0 = 0$

$$d(eE_{z\,\text{tr}})/dr_0 = 0, \quad d^2(eE_{z\,\text{tr}})/dr_0^2 < 0.$$

When $r_0 = a$, the loss is equal to zero.

4. The case of a charge which does not move along the axis allows an analysis of the radial force which acts on the charge. A charge moving in a channel with $\varepsilon_1\beta^2 < 1$, will radiate into the outer medium. The radiation reaction in non-central motion will be directed not only along the line of motion of the charge but also in the radial direction. From simple physical considerations, it is clear that if the radiation of the charge is directed into the external medium, the charge should experience a recoil force in the direction of the axis. However, in addition to the radiation reaction, there are also so-called "image" forces which arise as a result of the presence of the boundary. It is impossible to separate the effect of the image force from the reaction on the Cerenkov radiation.

We now find the conditions for which the charge is focused towards the axis. We write the radial component of the force which acts on the charge

$$F_r = eE_r + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]_r,$$

$$F_r = -\frac{e^2}{\pi v^2}\sum_{m=0}^{\infty} a_m \int \frac{\varkappa_1^3}{\varepsilon_1} I_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)$$

$$\times\left[K_m'\left(\frac{|\omega|}{v}\varkappa_1 r_0\right) + \lambda_{1m} I_m'\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)\right]\omega\, d\omega.$$

$$a_0 = 1,\ a_m = 2 \text{ for } m \neq 0$$

It is obvious that a contribution to the radial force is given only by terms with λ_{1m} under the integral. Hence,

$$F_r = -\frac{e^2}{\pi v^2}\sum_{m=0}^{\infty} a_m$$

$$\times \operatorname{Re}\int \frac{\varkappa_1^3}{\varepsilon_1}\lambda_{1m} I_m\left(\frac{|\omega|}{v}\varkappa_1 r_0\right) I_m'\left(\frac{|\omega|}{v}\varkappa_1 r_0\right)\omega\, d\omega. \tag{23}$$

The sign of F_r is determined by the sign of Re λ_{1m} since I_m and I_m' are positive real functions. As has been indicated above, λ_{1m} is a complicated function and in the general case it is impossible to ascertain the sign of the radial force. In the particular case of the emission of waves whose wavelength is much smaller than the radius of the channel, $|\omega|\varkappa_1 a/v \gg 1$, it is possible to make use of the asymptotic form of the Bessel function and the expression for λ_{1m} is simplified considerably.

For $|\omega|\varkappa_1 a/v \to \infty$:

$$\lambda_{1m} = \pi e^{-2|\omega|\varkappa_1 a/v}\,\frac{\varepsilon_1^2 t^2 - \varepsilon_2^2\varkappa_1^2 + 2i\varkappa_1 t\varepsilon_1\varepsilon_2}{\varepsilon_1^2 t^2 + \varepsilon_2^2\varkappa_1^2},$$

where $t^2 = \varepsilon_2\beta^2 - 1$ and $\varkappa_1^2 = 1 - \varepsilon_1\beta^2$. From the expression for λ_{1m} it is obvious that Re $\lambda_{1m} > 0$ when

$$\varepsilon_1^2 t^2 > \varepsilon_2^2\varkappa_1^2.$$

In this case the charge experiences a focusing force.

5. It is well known that when a charged particle moves uniformly in an infinite cylindrical channel in which the conditions for Cerenkov radiation are satisfied while the Cerenkov radiation cannot be excited in the medium external to the channel, *i.e.*, when $\varepsilon_1\beta^2 > 1$ and $\varepsilon_2\beta^2 < 1$, the spectrum of the radiation in the channel will exhibit a discrete character, in contrast to the continuous spectrum radiated by a particle in an unbounded medium.

The equation for the frequency spectrum has been obtained by Bolotovskii for the case in which the charged particle moves along the axis of the channel:[7]

$$\varepsilon_1 \sqrt{1-\varepsilon_2\beta^2}\, K_0\left(\frac{|\omega|}{v} a \sqrt{1-\varepsilon_2\beta^2}\right) \times J_0'\left(\frac{|\omega|}{v} a \sqrt{\varepsilon_1\beta^2-1}\right) + \varepsilon_2 \sqrt{\varepsilon_1\beta^2-1}\, J_0\left(\frac{|\omega|}{v} a \sqrt{\varepsilon_1\beta^2-1}\right) \times K_0'\left(\frac{|\omega|}{v} a \sqrt{1-\varepsilon_2\beta^2}\right) = 0 \quad (24)$$

In the short-wave range, *i.e.*, when $\omega\sqrt{1-\varepsilon\beta^2}\, a/v \gg 1$, Eq. (24) can be simplified considerably if the asymptotic expressions for the Bessel functions are used

$$\tan\left(\frac{|\omega|}{v} a \sqrt{\varepsilon_1\beta^2-1} - \frac{\pi}{4}\right) = -\frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\varepsilon_1\beta^2-1}{1-\varepsilon_2\beta^2}}, \quad (25)$$

this expression is similar to that derived by Frank for this case from simple physical considerations.[10]

For practical purposes it is of interest to analyze the radiation spectrum when a charged particle moves parallel to the axis of a cylindrical channel but at a distance r_0 from the axis. Using (16), the dispersion equation for this case can be written as follows:

$$s^2\varkappa_2^2\,[\varepsilon_1\varkappa_2 K_2 J_1' + \varepsilon_2 s J_1 K_2']\,[-sJ_1K_2' - \varkappa_2 J_1' K_2] + (mv/\omega a)^2 J_1^2 K_2^2 (\varepsilon_2-\varepsilon_1)^2\beta^2 = 0, \quad (26)$$

where for brevity, we have introduced the notation

$$s^2 = \varepsilon_1\beta^2 - 1, \quad \varkappa_2^2 = 1 - \varepsilon_2\beta^2,$$

$$K_2 = K_m\left(\frac{|\omega|}{v} a\varkappa_2\right), \quad J_1 = J_m\left(\frac{|\omega|}{v} as\right).$$

This equation has discrete roots. In the radiation of short waves, just as in (25), the asymptotic form of the Bessel function can be used and the dispersion equation can be written:

$$\tan\left(\frac{|\omega|}{v} as - \frac{m\pi}{2} - \frac{\pi}{4}\right) = \frac{-s\varkappa_2(\varepsilon_1+\varepsilon_2) + s\varkappa_2(\varepsilon_1-\varepsilon_2)\sqrt{1+4(mv/\omega a)^2\varepsilon_1/s^4\varkappa_2^2}}{2\varepsilon_1\varkappa_2^2}. \quad (27)$$

When $m = 0$, (27) goes over to (25). The expression in the right-hand part of (27) can be called the change of phase of the mth harmonic in total internal reflection. It is interesting to note that the equation which determines the eigen-frequency of the Cerenkov radiation produced by a particle moving with a velocity close to the velocity of light in the case where the external medium is a metal has exactly the same form as that pertaining to a Cerenkov counter, in which the radiation undergoes total internal reflection at the boundaries (for example, $\varepsilon_2 = 1$). Actually, the metal is characterized by $\varepsilon_2 \to -\infty$, $\varkappa_2 \to \infty$ while in the second case $\varepsilon_2 = 1$, $\beta = 1$, *i.e.*, $\varkappa_2 \to 0$. The dispersion equation (27) in both cases assumes the form

$$\tan\left(\frac{|\omega|}{v} as - \frac{m\pi}{2} - \frac{\pi}{4}\right) \to \infty,$$

$$i.e.,\ \frac{|\omega|}{v} as - \frac{m\pi}{2} + \frac{\pi}{4} = \pi n,$$

where $n = 0, 1, 2, \ldots$ Hence, the radiation spectrum produced by a parallel beam of relativistic particles is the same in a silvered and non-silvered Cerenkov counter.

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