Minami Transformation for Nucleon-Nucleon Scattering

R. M. RYNDIN AND IA. A. SMORODINSKII
United Institute for Nuclear Studies

(Submitted to JETP editor July 12, 1956)
J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 1200-1205 (May, 1957)

It is shown that for the case of nucleon-nucleon scattering there is no analog of the transformation of phases which, in the case of scattering of pions by nucleons, leaves the scattering cross section invariant. In the case of nucleon-nucleon scattering, a transformation consisting of a rotation of the spins does not yield a new set of phases which are physically meaningful.

INTRODUCTION

In THE CASE of scattering of spin ½ particles by spinless particles, there are two sets of phases[1-3] which cannot be distinguished except by means of polarization experiments (or by investigating the energy dependence of the scattering cross section at low energies). These sets of phases are obtained from one another by a simultaneous interchange of the phases referring to a given value of the total angular momentum. If δ_l^j denotes the phase of the scattered wave for given values of j and l, then the transformation is

$$\delta^j_{j-1/2} \rightleftarrows \delta^j_{j+1/2} \quad \text{(for all } j \text{ simultaneously)} \tag{I}$$

In a paper by the authors[4] it was shown that the existence of this transformation is a consequence of the obvious invariance of the cross section for scattering of an unpolarized beam with respect to rotation of the spin through an angle of $\pm\pi$ around the direction $\boldsymbol{\nu}$ of the wave vector of the scattered particles. The transformation (I) is easily generalized to the case of rotation of the spin through an arbitrary angle γ, and then takes the form:

$$\begin{aligned} E^j_{j-1/2} &\to i\cos\frac{\gamma}{2}E^j_{j-1/2} - \sin\frac{\gamma}{2}E^j_{j+1/2},\\ E^j_{j+1/2} &\to i\cos\frac{\gamma}{2}E^j_{j+1/2} - \sin\frac{\gamma}{2}E^j_{j-1/2}. \end{aligned} \tag{II}$$

E^j_l here denotes the quantity $\exp(2i\delta^j_l) - 1$. The existence of this transformation is proven most simply by applying to the scattering amplitude the operator $i\exp(-i\boldsymbol{\sigma\nu}\gamma/2)$, which is proportional to the operator for rotation of the spin through angle γ about the direction $\boldsymbol{\nu}$. The scattering cross section can be expressed in the form:*

$$d\sigma/d\omega = f^+_{1/2}f_{1/2},$$

where $f_{1/2}$ denotes the scattering amplitude when the projection of the total angular momentum along the direction of the incident beam is ½. It is obvious that the cross section is invariant under replacement of $f_{1/2}$ by $f'_{1/2} = i\exp(-i\boldsymbol{\sigma\nu}\gamma/2)f_{1/2}$. By using the standard rules for quantum-mechanical addition of angular momenta, the amplitude $f_{1/2}$ is expressed in terms of the eigenfunctions $Y^{1/2}_{j,l,1/2}(\boldsymbol{\nu})$ of the operators $\mathbf{j}^2$, $\mathbf{l}^2$, $\mathbf{s}^2$ and s_z:

$$f_{1/2} = \frac{\pi^{1/2}}{ik}\sum_{j=1/2}^{\infty}(j+1/2)^{1/2}\{E^j_{j-1/2}Y^{1/2}_{j,j-1/2,1/2}(\boldsymbol{\nu}) - E^j_{j+1/2}Y^{1/2}_{j,j+1/2,1/2}(\boldsymbol{\nu})\}. \tag{III}$$

If we now use the relation[5]

$$\boldsymbol{\sigma\nu}Y^{1/2}_{j,l,1/2}(\boldsymbol{\nu}) = Y^{1/2}_{j,l',1/2}(\boldsymbol{\nu}),$$

$$l' = 2j - l = \begin{cases} j+1/2 & \text{for } l = j-1/2\\ j-1/2 & \text{for } l = j+1/2,\end{cases}$$

the amplitude $f'_{1/2}$ can be put in the form

$$\begin{aligned} f'_{1/2} = \frac{\pi^{1/2}}{ik}\sum_{j=1/2}^{\infty}(j+1/2)^{1/2}\Big\{&\Big(i\cos\frac{\gamma}{2}E^j_{j-1/2} - \sin\frac{\gamma}{2}E^j_{j+1/2}\Big)Y^{1/2}_{j,j-1/2,1/2}(\boldsymbol{\nu})\\ &- \Big(i\cos\frac{\gamma}{2}E^j_{j+1/2} - \sin\frac{\gamma}{2}E^j_{j-1/2}\Big)Y^{1/2}_{j,j+1/2,1/2}(\boldsymbol{\nu})\Big\}. \end{aligned} \tag{IV}$$

Comparing (III) and (IV), we see that the amplitude $f'_{1/2}$ can be obtained from $f_{1/2}$ by means of the transformation (II), and consequently the cross section is in fact invariant under this transformation. The new

*As a consequence of the invariance of the interaction Hamiltonian with respect to rotations and reflections, the relation $f^+_{1/2}f_{1/2} = f^+_{-1/2}f_{-1/2}$. holds. Because of this the amplitude $f_{-1/2}$ does not appear explicitly in the expression for the scattering cross section for an unpolarized beam.

phases will be complex for all values of γ except the values $\gamma = \pm\pi$ considered by Minami. Therefore the transformation (II) with $\gamma \neq \pm\pi$ cannot be considered in the phase analysis of data on scattering of spin $\frac{1}{2}$ particles by spinless particles.

It is of interest to investigate the question of the existence of a similar type of invariance in the case of nucleon-nucleon scattering. We shall show that such a transformation does exist. However, this transformation does not satisfy the physical conditions of the problem. The symmetry of the system demands that the phases be independent of the magnetic quantum number m. But the phases obtained from the transformation we are considering turn out to depend on $|m|$. Consequently, in contradiction to our earlier expectation[4], there is no ambiguity of the phases of the type which occurs in a system with total spin $\frac{1}{2}$.

We shall limit ourselves to considering proton-proton scattering, since the more general case of neutron-proton scattering gives nothing essentially new.

1. AMPLITUDES AND DIFFERENTIAL SCATTERING CROSS SECTION

In this section we give a brief presentation of the method for describing the collision of identical particles with spin $\frac{1}{2}$, and express the amplitudes for proton-proton scattering in a form which is convenient for investigating their behavior under spin rotations.

The wave function of two protons moving along the z axis before the collision can be written in the center of mass system (c.m.s.) in the form

$$\psi_{sm} \sim (e^{ikz} - (-1)^{s+1}e^{-ikz})\chi_{sm} + F_{sm}(\nu)\, e^{ikr}/r. \tag{1.1}$$

Here χ_{sm} are spin functions ($s = 0$ and $s = 1$ in singlet and triplet states, respectively; m is the projection of the total spin on the z axis), k is the wave number, while $\boldsymbol{\nu}$ is a unit vector along the direction of motion of the scattered particles.

The scalar product* $\langle F_{sm}, F_{sm}\rangle$ determines the differential scattering cross section $d\sigma_{sm}(\boldsymbol{\nu})$ for given initial spin s and spin projection m. The cross section for scattering of an unpolarized beam is gotten by averaging the cross section $d\sigma_{sm}(\boldsymbol{\nu})$ over all possible initial spin states:

$$d\sigma(\nu)/d\omega = {}^1/_4\Big\{\langle F_{00}(\nu),\ F_{00}(\nu)\rangle + \sum_{m=-1}^{+1}\langle F_{1m}(\nu),\ F_{1m}(\nu)\rangle\Big\}. \tag{1.2}$$

Using the standard methods for investigating the asymptotic behavior of wave functions (*cf.* for example, the paper of Blatt and Biedenharn[6]), one can easily obtain explicit expressions for the amplitudes $F_{sm}(\boldsymbol{\nu})$ in terms of the elements of the scattering matrix S and the functions $Y^m_{j,l,s}$. The singlet scattering in a state with given j is described by means of the single matrix element $S_j^j = \exp(2i\Delta_j)$, where Δ_j denotes the corresponding phase. The description of the triplet scattering is much more complicated. Here we have to distinguish two groups of states, with $j = l$ and $j \pm 1 = l$ respectively. In the first group ($j = l$) are the states with odd j, while the second contains states with even j. In the states of the first group, as in the singlet states, the orbital angular momentum is conserved, so that the scattering is again described by a single matrix element $T_j^j = \exp(2i\delta_j)$. In states of the second group the orbital angular momentum is not conserved, and transitions are possible between states with $l = j + 1$ and $l' = j - 1$ (*e.g.*, the states ${}^3P_2$ and ${}^3F_2$). Consequently the scattering in states with a given j is described by a two-by-two matrix. In the example of the states ${}^3P_2$ and ${}^3F_2$, its matrix elements are given by the transition probabilities ${}^3P_2 \to {}^3P_2$, ${}^3P_2 \to {}^3F_2$, ${}^3F_2 \to {}^3P_2$, and ${}^3F_2 \to {}^3F_2$. This matrix is symmetric (the equality of the transition amplitudes for ${}^3P_2 \to {}^3F_2$ and ${}^3F_2 \to {}^3P_2$ is a consequence of the invariance of the interaction Hamiltonian under time reversal), and is unitary. We shall label the matrix elements of the scattering matrix in states with a given j according to the scheme:

$l \setminus l'$	$j-1$	$j+1$
$j-1$	A_j'	C_j
$j+1$	C_j	B_j'

The unitarity condition enables us to express[6,7] the three complex parameters A_j', B_j', and C_j in terms of three independent real parameters consisting of the two phases δ_j^{I} and δ_j^{II} and a mixing parameter ε_j, by means of the relations:

---

* By the scalar product of two linear combinations of spin functions

$$a = \sum_{s'}\sum_{m'} a_{s'm'}\chi_{s'm'} \quad \text{and} \quad b = \sum_{s''}\sum_{m''} b_{s''m''}\chi_{s''m''}$$

we mean the quantity

$$\langle a, b\rangle = \sum_s\sum_m a^*_{sm} b_{sm}.$$

$$
\begin{aligned}
A'_j &= \cos^2 \varepsilon_j \exp(2i\delta_j^{\mathrm{I}}) + \sin^2 \varepsilon_j \exp(2i\delta_j^{\mathrm{II}}),\\
B'_j &= \sin^2 \varepsilon_j \exp(2i\delta_j^{\mathrm{I}}) + \cos^2 \varepsilon_j \exp(2i\delta_j^{\mathrm{II}}), \quad (1.3)\\
C_j &= {}^1/_2 \sin 2\varepsilon_j (\exp(2i\delta_j^{\mathrm{I}}) - \exp(2i\delta_j^{\mathrm{II}})).
\end{aligned}
$$

The mixing parameter is associated with the tensor forces and vanishes if they are absent.

The scattering amplitudes are given by:

$$
\begin{aligned}
F_{00}(\nu) &= \frac{(4\pi)^{1/2}}{ik} \sum_j (2j+1)^{1/2} S_j Y^0_{j,j,0}(\nu),\\
F_{1,1}(\nu) &= \frac{(2\pi)^{1/2}}{ik} \sum_j \{[(j+1)^{1/2} A_j + j^{1/2} C_j]\, Y^1_{j,j-1,1}(\nu) - (2j+1)^{1/2}\, T_j Y^1_{j,j,1}(\nu)\\
&\quad + [j^{1/2} B_j + (j+1)^{1/2} C_j]\, Y^1_{j,j+1,1}(\nu)\},\\
F_{1,0}(\nu) &= \frac{(4\pi)^{1/2}}{ik} \sum_j \{[j^{1/2} A_j - (j+1)^{1/2} C_j]\, Y^0_{j,j-1,1}(\nu) +\\
&\quad + [-(j+1)^{1/2} B_j + j^{1/2} C_j]\, Y^0_{j,j+1,1}(\nu)\}, \qquad (1.4)\\
F_{1,-1}(\nu) &= \frac{(2\pi)^{1/2}}{ik} \sum_j \{[(j+1)^{1/2} A_j + j^{1/2} C_j]\, Y^{-1}_{j,j-1,1}(\nu) +\\
&\quad + (2j+1)^{1/2} T_j Y^{-1}_{j,j,1}(\nu) + [j^{1/2} B_j + (j+1)^{1/2} C_j]\, Y^{-1}_{j,j+1,1}(\nu)\}.
\end{aligned}
$$

S_j, T_j, A_j and B_j denote the quantities $S_j' - 1$, $T_j' - 1$, $A_j' - 1$ and $B_j' - 1$. The summation in the singlet scattering amplitude is taken only over even j in accordance with the Pauli principle. In the triplet scattering amplitude, the summation is taken over odd j for states with $j = l$ and over even j for states with $j \pm 1 = l$.

2. BEHAVIOR OF SCATTERING AMPLITUDES UNDER SPIN ROTATIONS

We now proceed to investigate the transformation of the scattering amplitudes $F_{sm}(\nu)$ under rotations of the total spin and the spins of the individual protons, and find substitutions on the elements of the scattering matrix which lead to the same transformations of the amplitudes.

We start by studying rotations of the total spin $s = (\sigma_1 + \sigma_2)/2$, where the σ_i are the Pauli spin operators for the two protons. The operator for rotation of the spin s through angle γ around the axis ν is given by the direct product of the operators for rotation of the proton spins through the same angle:

$$
\begin{aligned}
R(\nu, \gamma) = \exp(-is\nu\gamma) &= \cos^2 \frac{\gamma}{2}\\
- is\nu \sin\gamma &+ \sin^2 \frac{\gamma}{2}(1 - 2(s\nu)^2). \qquad (2.1)
\end{aligned}
$$

The effect of $R(\nu, \gamma)$ on singlet spin functions is the same as that of the unit operator, since the projection of the total spin on any direction is zero in the singlet state.

We note first of all that, in the case of identical particles (two protons), rotations of the total spin s through an arbitrary angle γ cannot lead to physically admissible substitutions on the elements of the scattering matrix, since the amplitudes $F'_{sm} = R(\nu, \gamma) F_{sm}$ which result from this transformation do not satisfy the Pauli principle. On the one hand the rotation operator $R(\nu, \gamma)$ commutes with the square of the total spin s^2, and consequently leaves the spin parity of the state unchanged. On the other hand, rotation through an arbitrary angle γ changes a state with a definite orbital parity into a linear combination of states with different parities. The exception is the case of $\gamma = \pm\pi$, when the rotation operator becomes $1 - 2(s\cdot\nu)^2$, which does not change the parity of the state.

One can show that the amplitudes $F'_{sm}(\nu)$, corresponding to a total spin s rotated through the angle $\pm\pi$, have the form

$$
\begin{aligned}
F'_{0,0}(\nu) &= R(\nu, \pm\pi) F_{0,0}(\nu) = F_{0,0}(\nu),\\
F'_{1,1}(\nu) &= R(\nu, \pm\pi) F_{1,1}(\nu) = \frac{(2\pi)^{1/2}}{ik} \sum_j \Big\{ -\frac{1}{2j+1} [(j+1)^{1/2} A_j + j^{1/2}(2j+3)\, C_j +
\end{aligned}
$$

$$
\begin{aligned}
&+ (j+1)^{1/2} 2jB_j]\, Y^1_{j,j-1,1}(\nu) + (2j+1)^{1/2} T_j Y^1_{j,j,1}(\nu) - \frac{1}{2j+1}[j^{1/2} 2(j+1)A_j \\
&\qquad + (j+1)^{1/2}(2j-1)C_j - j^{1/2}B_j]\, Y^1_{j,j+1,1}(\nu)\Big\}, \\
&F'_{1,0}(\nu) = R(\nu, \pm\pi)\, F_{1,0}(\nu) = \frac{(4\pi)^{1/2}}{ik}\sum_j \Big\{-\frac{1}{2j+1}[j^{1/2}A_j + (j+1)^{1/2}(2j-1)C_j \\
&\qquad - j^{1/2} 2(j+1)B_j]\, Y^0_{j,j-1,1}(\nu) - \frac{1}{2j+1}[(j+1)^{1/2} 2jA_j - j^{1/2}(2j+3)C_j \\
&\qquad + (j+1)^{1/2}B_j]\, Y^0_{j,j+1,1}(\nu)\Big\}, \qquad\qquad (2.2)\\
&F'_{1,-1}(\nu) = R(\nu, \pm\pi)\, F_{1,-1}(\nu) = \frac{(2\pi)^{1/2}}{ik}\sum_j \Big\{-\frac{1}{2j+1}[(j+1)^{1/2}A_j + j^{1/2}(2j+3)C_j \\
&+ (j+1)^{1/2} 2jB_j]\, Y^{-1}_{j,j-1,1}(\nu) - (2j+1)^{1/2} T_j Y^{-1}_{j,j,1}(\nu) - \frac{1}{2j+1}[j^{1/2} 2(j+1)A_j \\
&\qquad + (j+1)^{1/2}(2j-1)C_j - j^{1/2}B_j]\, Y^{-1}_{j,j+1,1}(\nu)\Big\}.
\end{aligned}
$$

This transformation of the amplitudes is equivalent to their transformation by the following substitutions on the triplet elements of the scattering matrix:

$$
m = \pm 1 \left\{
\begin{aligned}
&(j+1)^{1/2}A_j + j^{1/2}C_j \to \\
&\to -\frac{1}{2j+1}[(j+1)^{1/2}A_j + j^{1/2}(2j+3)C_j + (j+1)^{1/2}2jB_j], \\
&j^{1/2}B_j + (j+1)^{1/2}C_j \to \\
&\to -\frac{1}{2j+1}[j^{1/2}2(j+1)A_j + (j+1)^{1/2}(2j-1)C_j - j^{1/2}B_j], \\
&T_j \to -T_j,
\end{aligned}
\right. \qquad (2.3a)
$$

$$
m = 0 \left\{
\begin{aligned}
&j^{1/2}A_j - (j+1)^{1/2}C_j \to \\
&\to -\frac{1}{2j+1}[j^{1/2}A_j + (j+1)^{1/2}(2j-1)C_j - j^{1/2}2(j+1)B_j], \\
&-(j+1)^{1/2}B_j + j^{1/2}C_j \to \\
&\to -\frac{1}{2j+1}[(j+1)^{1/2}2jA_j - j^{1/2}(2j+3)C_j + (j+1)^{1/2}B_j], \\
&T_j \to -T_j.
\end{aligned}
\right. \qquad (2.3b)
$$

The cross section $d\sigma(\nu)$ remains invariant when we replace $F_{sm}(\nu)$ by $F'_{sm}(\nu) = R(\nu, \pm\pi)F_{sm}(\nu)$ because of the unitarity of the operator $R(\nu, \pm\pi)$. This means that $d\sigma(\nu)$ is also invariant with respect to the substitutions (2.3). However, substitution (2.3a), which gives the desired transformation of the amplitudes $F_{1,\pm1}(\nu)$ differs from (2.3b), which gives the transformation of $F_{1,0}(\nu)$. The substitutions do not depend on the sign of the projection m of the total angular momentum, but they do depend on its absolute value. The difference between the substitutions (2.3a) and (2.3b) is seen most simply in the case where tensor forces are absent ($C_j = 0$; this does not result in disappearance of polarization effects, since a nuclear spin-orbit interaction between nucleons may still be present). In this case the substitutions (2.3) become a determined set of substitutions on a pair of parameters and take the form:

$$
\begin{aligned}
m = \pm 1:\ A_j &\to -\frac{1}{2j+1}[A_j + 2jB_j], \\
B_j &\to -\frac{1}{2j+1}[2(j+1)A_j - B_j], \\
T_j &\to -T_j; \qquad (2.4a)
\end{aligned}
$$

$$
\begin{aligned}
m = 0:\ A_j &\to -\frac{1}{2j+1}[A_j - 2(j+1)B_j], \\
B_j &\to +\frac{1}{2j+1}[2jA_j + B_j], \quad T_j \to -T_j. \qquad (2.4b)
\end{aligned}
$$

The difference between (2.4a) and (2.4b) is obvious. The reason for this difference is the difference in the weights with which the angular wave functions $Y^m_{j,l,s}$ with different values of $|m|$ enter into the expansion of the initial plane waves [the first term in (1.1)]. The distinction between the substitutions (2.3a) and (2.3b) in the general case follows directly from the incompatibility of the system of four equations for the three elements of the new scattering matrix S', which do not depend on the value of the projection. These equations are obtained from (2.3a) and (2.3b) by replacing the $\rightarrow$ by an equality sign and replacing A_j, B_j and C_j on the left sides of the equations by the elements $S' - 1$.

The difference in the substitutions corresponding to different values of $|m|$ means that the elements of the scattering matrix become dependent on $|m|$ after we carry out the indicated substitution. This fact is not compatible with the symmetry of the system and consequently we cannot, by means of rotations of the total spin, construct a physically admissible set of scattering matrix elements from the set obtained by analysis of data on the scattering of unpolarized protons.

We can attack the problem of investigating the transformation of the scattering amplitudes under rotations of the spin of one of the protons by a similar method. These transformations, like the amplitude transformations for rotations of the total spin, do not give physically admissible substitutions on the elements of the scattering matrix. In the first place, the amplitude resulting from the transformation by a rotation of one of the spins corresponds to a scattering matrix with non-zero elements for singlet-triplet transitions (the operator $\sigma_i \cdot \nu$ does not commute with the square of the total spin $\mathbf{s}^2$), and consequently does not satisfy the Pauli principle. Secondly, the substitutions leading to the required amplitude transformation depend on $|m|$, as in the case of rotations of the total spin. However, in the special case when only $^1S_0$ and $^3P_0$ states (with zero projection of the total angular momentum) contribute to the scattering, we find the well-known invariance of the cross section with respect to interchange of the phases $\delta(^1S_0)$ and $\delta(^3P_0)$, which occurs for a rotation of the spin of one of the protons through $\pm\pi$.

Finally, we note that a treatment of the collision of non-identical particles, for which the limitations imposed by the Pauli principle are absent, gives nothing new beyond what was found for proton-proton scattering. Moreover, the results of the investigation of the behavior of the scattering amplitudes under rotation of the total spin of the two-nucleon system can be taken over directly to the case of scattering of deuterons by spin zero particles.

The only system (for the case of non-relativistic particles), for which the spin rotation leads to an ambiguity of phases in the analysis of data on scattering of unpolarized particles is the system with total spin ½, since in this case there is only value of $|m|$.

[1] S. Minami, Progr. Theor. Phys. **11**, 213 (1954).

[2] S. Hayakawa *et al.*, Progr. Theor. Phys. **11**, 332 (1954).

[3] N. Fukuda *et al.*, Progr. Theor. Phys. **12**, 79 (1954).

[4] R. M. Ryndin and Ia. A. Smorodinskii, Dokl. Akad. Nauk SSSR **103**, 69 (1955).

[5] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics*, Moscow, 1953, p. 67; H. Bethe and F. de Hoffman, Mesons and Fields, Vol. II, New York, 1955, p. 72.

[6] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

[7] J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952).

Translated by M. Hamermesh
241