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The Thermodynamics of Liquid He^3

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The thermodynamics of liquid He^3 is considered on the basis of the Fermi liquid model proposed by L. Landau. The entropy, the specific heat, and the magnetic susceptibility of liquid He^3 are computed for the possible types of the energy spectrum of the liquid. Comparison with experiment is carried out.

LET US CONSIDER the thermodynamics of liquid helium on the basis of the Fermi liquid model proposed by Landau[1]. According to Ref. 1, for small deviations of the distribution function from its equilibrium value at $T = 0$, the excitation energy may be presented in the form

$$\begin{aligned} \varepsilon &= \varepsilon(p) + \int f(p, p')\,\nu(p')\,d\tau' \quad d\tau \\ &= g\,dp_x dp_y dp_z / (2\pi\hbar)^3, \end{aligned} \tag{1}$$

where $\nu = n - n_0$, and g is the statistical weight (here ε does not depend on the spin). In the usually-employed perfect-gas model, the excitation energy $\varepsilon(p)$ is written in the form $p^2/2m$, where m is a certain effective mass. We will see later, however, that such a form of the spectrum leads to results which, although not in contradiction with the experimental data, do not agree with them too well. It is why we will consider for $\varepsilon(p)$ another function proposed by L. D. Landau, namely:

$$\varepsilon(p) = (p - p_0)^2/2m, \tag{2}$$

where m is the effective mass. In this case, for $T = 0$, the distribution function in momentum space will be not a sphere but a spherical shell. We will assume the thickness Δ of this shell as small with respect to its radius p_0 ("bubble"); rigorously speaking, in the opposite case Eq. (2) would be erroneous and $\varepsilon(p)$ could not be considered as a symmetric function with respect to the minimum point $p = p_0$. In the case of a spectrum of type (2), we are faced with a unique situation where the Fermi-liquid theory is only valid for such temperatures, for which the deviations from the zeroth distribution function are small. For the perfect-gas model this obviously corresponds to temperatures $T \ll T_D$, where T_D is the temperature of degeneracy. In the case of a spectrum of type (2), however, even for temperatures considerably higher than the temperature of degeneracy and which correspond to a Boltzmann distribution, the distribution function continues to decrease as one gets away from $p = p_0$, so that effectively all the excitations will be localized in the neighborhood of that point. One sees therefore that there exists a temperature region $T > T_D$ for which the deviation from the zeroth function is negligible. This makes possible in this case evaluation of the thermodynamical functions in the Fermi as well as in the Boltzmann regions.

As we will see, a spectrum of type (2) agrees with the experimental data better than the perfect gas model, but not so much as to exclude the possibility of this model for He^3, more so because it appears to be more natural.

1. *Specific Heat and Entropy.* For a non-zero temperature, the distribution function is the usual Fermi function

$$n = [e^{(\varepsilon - \mu)/kT} + 1]^{-1}$$

(μ — chemical potential). Let us compute the specific heat in the case of a spectrum of the type (2). We start from the expression for the energy of the system

$$E = \int \varepsilon n\,d\tau = \frac{4\pi p_0^2 g \sqrt{2m}}{(2\pi\hbar)^3} (kT)^{3/2} I_{1/2}, \tag{3}$$

where*

$$I_\alpha = \int\limits_0^\infty \frac{x^\alpha dx}{(e^x / A) + 1}, \quad A = e^{\mu/kT}.$$

Further, differentiating this expression with respect to temperature, and using the condition $N = \text{const.}$, we find the specific heat per particle:

$$c = \frac{1}{N}\left(\frac{dE}{dT}\right)_N = k\left[\frac{3}{2}\frac{I_{1/2}}{I_{-1/2}} - \frac{1}{4}\frac{I_{-1/2}}{(\partial I_{1/2} / \partial A)}\right]. \tag{4}$$

In this fashion, the specific heat is expressed in

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* Detailed tables of the integrals I_α are available (Ref. 2).

terms of the parameter A which, in turn, is expressed in terms of the temperature, making use of the relationship

$$N = \int n d\tau = \frac{4\pi p_0^2 g \sqrt{2m}}{(2\pi\hbar)^3} (kT)^{1/2} I_{-1/2}. \qquad (5)$$

Let us also find the formula for the entropy. We start from the definition of the entropy

$$S = -k \int \{ n \ln n + (1-n) \ln (1-n) \} d\tau. \qquad (6)$$

Substituting here the expression for the distribution function, we get after simple integration by parts

$$s = S/N = k \{ (3 I_{1/2} / 2 I_{-1/2}) - \ln A \}. \qquad (7)$$

Let us use the known expansion of the Fermi-type integrals I_α and find the asymptotic expressions for c and s at low temperatures:

$$c = s = \gamma T;$$

$$\gamma = g^2 p_0^4 m k^2 / 3\pi^2 \hbar^6 N^2 = 4\pi^2 m k^2 / 3\Delta^2. \qquad (8)$$

Analogous formulas correspond to the perfect gas model, for which

$$\gamma = (m / 2\hbar^2) (\sqrt{2}\pi g / 3N)^{2/3} k^2 = \pi^2 m p_{\mathrm{gr}}^{-2} k^2. \qquad (9)$$

Comparing formulae (8) and (9) with the experimental data on the entropy of He^3 at low temperatures[3], we can find the parameter γ and therefrom the parameters of the spectrum which occur in it. Unfortunately, at the present time, the experimental curve is extended only to the beginning of the linear region. On the basis of these data, it is possible to obtain an approximate value for the parameter γ. It turns out to be equal to $\gamma = 3$ cal/mol.deg.

In the perfect gas model the effective mass m turns out to be equal to $m = 1.43\, m_3$ (m_3 is the mass of the He^3 atom). The limiting momentum p_{lim} can be found from the total number of particles ($\rho = 0.078$ g/cm³) $(p_{\mathrm{lim}}/\hbar) = (3\pi^2 N)^{1/3} = 0.76 \cdot 10^8$ cm^{-1}. The limiting Fermi energy is equal to $\mu/k = \varepsilon_{\mathrm{lim}}/k = p_{\mathrm{lim}}^2/2mk = 3.3°$K. As to the parameters characterizing a spectrum of type (2), the situation is somewhat more complicated, since all the expressions for the thermodynamical observables depend on the sole combination of parameters $p_0^2\sqrt{2m}$. This is why we find from the experimental data on the specific heat only this combination $(p_0/\hbar)^2\sqrt{m/m_3} = 2 \times 10^{15}$ cm^{2}. In this case, as the momentum p_0 has to be larger than the momentum of the Fermi sphere, one can determine for the effective mass of this spectrum the inequality $m < 0.12 m_3$. No one should be surprised by the smallness of this effective mass. The effective mass for a spectrum of such a type does not have to be close to the mass of the particles of the liquid (let us recall, for instance, that the mass of a roton in liquid He^4 is close to the proton mass ($0.26 m_4$, m_4 – mass of He^4)). Let us express the number of particles through the limiting energy of such a spectrum, $N = \sqrt{\varepsilon_{\mathrm{lim}}}\, 8\pi p_0^2 g \sqrt{2m}/(2\pi\hbar)^3$. This yields the value of the limiting energy $\varepsilon_{\mathrm{lim}} = 1.05°$ K.

The linear law for the specific heat is observed experimentally only for temperatures of the order of 0.2°K. It seems to us that a spectrum of type (2) corresponds better to this situation because—for this spectrum—the temperature of degeneracy is equal to 1°K. It also seems that a spectrum of type (1) gives a temperature of degeneracy which is too high (3.3°K). If the temperature exceeds the temperature of degeneracy, the formulae of the Boltzmann statistics may be applied (in this case, the parameter A tends to zero). For the perfect-gas model, one obviously gets $c = 3k/2$. This law does not correspond to experiment; this shows, however, only the inapplicability of such a model above T_D

For a type (2) spectrum, $c \to k/2$ as A tends to zero. In this region, therefore, the specific heat of He^3 tends to $k/2$, or, calculated per mole, to 1 calorie. This result is absolutely natural because the law of equipartition applies at these temperatures, and for this type of spectrum there is only one degree of freedom $(p - p_0)$.

A characteristic property of such a spectrum is, therefore, that, above the temperature of degeneracy, the specific heat tends to $k/2$ and not to $3k/2$, as it does in the case of the perfect-gas model.

We obtain therefore the following qualitative picture for the temperature dependence of the specific heat in the case of a type (2) spectrum. At low temperature (below T_D), there is a linear low; above T_D, c tends to the constant magnitude 1 cal/mol. deg. (there is a very weak maximum in the neighborhood of T_D). The result of the calculations for the value

$$(p_0^2/\hbar^2)\sqrt{m/m_3} = 2 \cdot 10^{15}\ \mathrm{cm}^{-2}.$$

are shown on Fig. 1. The qualitative and quantitative agreement of the theory and of the experimental

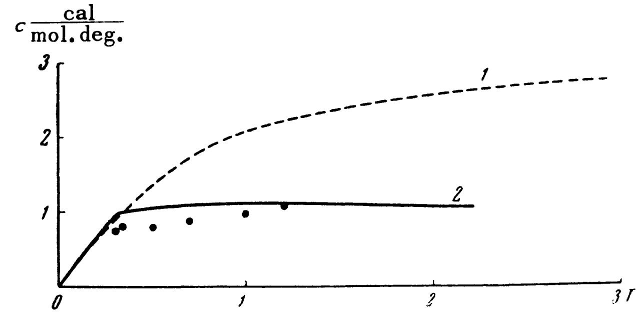

FIG. 1. Specific heat of liquid He^3: 1 — perfect gas model, 2 — spectrum of type (2), points — experimental data.[3]

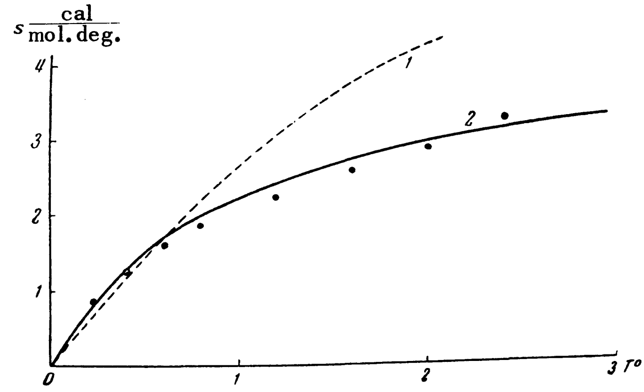

FIG. 2. Entropy of liquid He^3: 1 — perfect gas model, 2 — spectrum of type (2), points — experimental data[3].

data constitutes some evidence in favor of a type (2) spectrum for He^3.

We want to emphasize the existence of a large region of the experimental curve (about from 0.5 to 1.5°K), where the specific heat is approximately equal to 1 cal/mol. deg. The perfect-gas model cannot, of course, give a picture of such a kind. In this model the specific heat tends to 3 cal/mol.deg fairly rapidly, as the temperature increases. But one has to remark that the comparison of the experimental data with the results obtained from this model become, as already mentioned, forbidden as the temperature approaches the temperature of degeneracy. However, even away from the temperature of degeneracy, in the region of 1°K, the experimental data are twice as small as the theoretical results for the perfect-gas model with the parameters we chose.

The curves for the entropy are shown on Fig. 2.

2. *Magnetic Susceptibility.* Let us recall the equations of the magnetic susceptibility[1]. Let us assume that the spin interaction is of pure exchange type. With this assumption, let us separate from the function which characterizes the dependence of the excitation energy on the distribution function that part which depends on the spin interaction: $f = \varphi + \psi(s, s')$. According to Ref. 1, the magnetic susceptibility is

$$\chi = -\frac{\mu^2}{4}\int \frac{\partial n}{\partial \varepsilon} d\tau \Big/ \left[1 - \frac{\bar{\psi}}{4}\int \frac{\partial n}{\partial \varepsilon} d\tau\right], \tag{10}$$

where μ is the magnetic moment of the He^3 atom and $\bar{\psi}$ is the ψ function averaged over angles for momenta corresponding to the Fermi limit.

Using the notation defined above, let us rewrite this formula in the form

$$\frac{\mu^2}{\chi} = \bar{\psi} - 4\Big/\int \frac{\partial n}{\partial \varepsilon} d\tau = \bar{\psi} + 4/a\,(kT)^{\alpha}(\partial I_{\alpha}/\partial A)$$

$$\left.\begin{aligned} a &= 4\pi g m \sqrt{m}/(2\pi\hbar)^3, \\ \alpha &= {}^1/_2 \quad \text{(perfect gas model)} \\ a &= 4\pi p_0^2\, g \sqrt{2m}/(2\pi\hbar)^3, \\ \alpha &= -{}^1/_2 \quad \text{[spectrum (2)]} \end{aligned}\right. \tag{11}$$

In the low-temperature limit, the latter formula takes the form

$$\frac{\mu^2}{\chi} = \overline{\psi} + \frac{4}{a}\left(N\,\frac{1+a}{a}\right)^{-\alpha/(\alpha+1)}.$$

Therefore, in the low temperature region (below T_D) the magnetic susceptibility of the Fermi liquid tends to a constant limit.

In the high temperature region, the asymptotic law for χ has the form $\mu^2/\chi_\infty T = 4k/N$. Usually the quantity $1/(\chi/\chi_\infty T)$ is used. In the low temperature region, this quantity is equal to

$$\frac{1}{(\chi/\chi_\infty T)} = \overline{\psi}\,\frac{N}{4k} + \frac{\pi^2 k}{3\gamma}, \tag{12}$$

where γ is determined by Eq. (8) or (9).

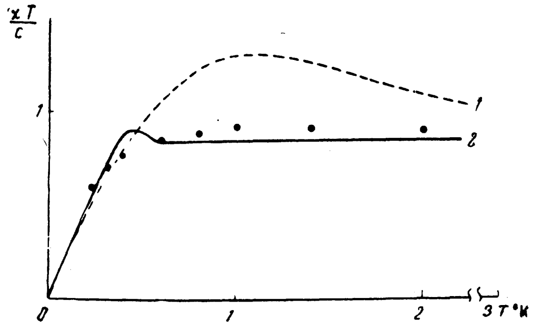

FIG. 3. Magnetic susceptibility of liquid He^3: 1 – perfect gas model, 2 – spectrum of type (2), points – experimental data[4].

From the comparison of Eq. (12) with experimental data[4], one gets for $\psi N/4k$ the value –1.05. This magnitude constitutes about 2/3 of the second term in (12). In this fashion, the exchange interaction plays a considerable role in the magnitude of the magnetic susceptibility of liquid He^3. The sign of the effect is such that the exchange interaction favors, in this case, a parallel orientation of the spins. However, this interaction does not yield ferro-magnetism, because the Fermi tendency for an anti-parallel orientation of the spin prevails. It is possible that with increasing pressure the parameters will change in such a way that ferromagnetism will be possible; however, so far, there are no experimental indications for such a possibility.

The curves for the susceptibility are shown on Fig. 3. (On this figure, c is the normalized Curie constant.) One sees from the curves that, here too, a type (2) spectrum agrees better with experimental data.

In conclusion, we wish to thank Academician L. D. Landau for useful advice and discussion of our results.

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