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The Energy Spectrum of Cascade Electrons in Lead

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Recursion formulas have been obtained for the nth moment of the distribution function of a shower containing particles with energies above a prescribed value. The first two moments of the electron distribution function in lead have been calculated. The moment method has been applied to compute the energy spectrum of cascade electrons in lead.

THE method of moments[1-7] has been applied to obtain a series of important characteristics of electron-photon cascade showers in heavy elements. This method is based on the computation of the distribution function $N(E_0, 0, t)$ at a depth t for the total number of particles in a shower initiated by a primary particle of energy E_0, by means of recursion formulas[2] for the moments $\overline{t^n}(E_0, 0)$. The moments are defined by the expression

$$\overline{t^n}(E_0, 0) = \int\limits_0^\infty N(E_0, 0, t)\, t^n dt \Big/ \int\limits_0^\infty N(E_0, 0, t)\, dt.$$

In Ref. 3 we suggested a method for computing cascade curves, including the energy dependence of the total absorption coefficient of photons, $\sigma(E)$, and the Rutherford scattering of charged particles, by means of a set of polynomials that are orthogonal in the interval $(0, \infty)$. It was shown in Ref. 3 that the use of the first two moments $\overline{t}$ and $\overline{t^2}$ allows one to compute cascade curves $N(E_0, 0, t)$ for heavy and light elements differing by less than 5 to 10% from the exact curves over a primary energy range of 0.1β to $\sim 1000\beta$. This method is easily generalized to allow computation of the energy spectrum of cascade particles in heavy elements. Carrying out a computation similar to that of Belen'kii and Maksimov,[4] one obtains recursion formulas for the moments of the distribution in depth for particles of energy greater than E^0, in showers initiated by primary particles of energy E_0:

$$\{\overline{t_p^n}(E_0, E^0)\}^p = n\int\limits_{E^0}^{E_0}\Big[\{\overline{t_p^{n-1}}(E', E^0)\}^p \int\limits_{E^0}^{E'} P_{p,0,0}$$

$$\times (E_0, E)\, dE \cdot P_{p,0,n}(E_0, E')$$

$$+ \{\overline{t_p^{n-1}}(E', E^0)\}^\Gamma \int\limits_{E^0}^{E_0} P_{\Gamma,0,0}(E_0, E)\, dE \cdot \Gamma_{p,0,n}$$

$$\times (E_0, E')\Big]\, dE' \Big/ \int\limits_{E^0}^{E_0} P_{p,0,0}(E_0, E)\, dE,$$

$$\{\overline{t_p^n}(E_0, E^0)\}^\Gamma = n\int\limits_{E^0}^{E_0}\Big[\{\overline{t_p^{n-1}}(E', E^0)\}^p \int\limits_{E^0}^{E'} P_{p,0,0}$$

$$(E_0, E)\, dE \cdot P_{\Gamma,0,n}(E_0, E')$$

$$+ \{\overline{t_p^{n-1}}(E', E^0)\}^\Gamma \int\limits_{E^0}^{E'} P_{\Gamma,0,0}(E_0, E)\, dE \tag{1}$$

$$\times\ \Gamma_{\Gamma,0,n}(E_0, E')\Big]\, dE' \Big/ \int\limits_{E^0}^{E_0} P_{\Gamma,0,0}(E_0, E)\, dE.$$

The upper indices p and Γ denote showers initiated respectively by primary electrons and photons of energy E_0, $P_{p,0,0}(E_0,E) = \int\limits_0^\infty \int\limits_{(\omega)} P_p(t, E, \theta)\, d\omega dt$, and $\Gamma_{p,0,0}(E_0, E) = \int\limits_0^\infty \int\limits_{(\omega)} \Gamma_p(t, E, \theta) d\omega dt$ are the "equilibrium" spectra for electrons and photons in showers caused by primary electrons; $P_p(t, E, \theta)$ and $\Gamma_p(t, E, \theta)$ are the distribution functions for electrons and photons at a depth $\overline{t}$, energy E, and angle θ in showers due to primary electrons. Analogous recursion formulas are easily obtained for the moments of the photon distribution function.

We list in Table I the moments $\overline{t}$ and $\overline{t^2}$ computed from Eq. (1) for various values of E_0 and E^0 including the energy dependence of the total absorption coefficient of photons $\sigma(E)$ (the dependence of $\sigma(E)$ upon E was taken from Ref. 8, and the Rutherford scattering of charged particles. In carrying out

TABLE I.

$\varepsilon_0 = 50$			$\varepsilon_0 = 90$			$\varepsilon_0 = 118{,}1$		
ε^0	$\{\overline{t}_p(\varepsilon_0, \varepsilon^0)\}^\Gamma$	$\{\overline{t^2_p}(\varepsilon_0, \varepsilon^0)\}^\Gamma$	ε^0	$\{\overline{t}_p(\varepsilon_0, \varepsilon^0)\}^\Gamma$	$\{\overline{t^2_p}(\varepsilon_0, \varepsilon^0)\}^\Gamma$	ε^0	$\{\overline{t}_p(\varepsilon_0, \varepsilon^0)\}^\Gamma$	$\{\overline{t^2_p}(\varepsilon_0, \varepsilon^0)\}^\Gamma$
0	6,30	58,5	0	7,08	70,3	0	7,43	74,6
0,183	6,12	54,9	0,183	6,89	66,6	0,183	7,25	70,9
0,738	5,59	45,4	0,932	6,22	53,6	0,838	6,57	59,5
1,292	5,18	40,2	1,680	5,74	45,4	1,493	6,12	50,8
2,402	4,66	31,8	2,428	5,40	38,9	2,476	5,67	43,9
3,511	4,30	26,7	4,674	4,79	31,6	4,114	5,19	36,3
5,175	3,90	22,0	6,919	4,33	27,5	6,734	4,54	29,1
6,839	3,61	18,6						

the numerical computations, we have used the expression for the "equilibrium" spectrum including scattering, $P_{p,0,0}(E_0, E), \ldots$, derived in Ref. 4. In that reference, the expressions for the "equilibrium" spectrum were obtained from an approximate solution, integrated over depth, of the basic equations of cascade theory including scattering. This approximation was shown in Ref. 5 to be within 6% of the exact value. Note that the expression for the "equilibrium" spectrum appears as the zero moment in the recursion formulas (1). An error in the zero moment will give rise to a growing error in higher moments, so that in adopting the moment method, it is important to have an accurate expression for the "equilibrium" spectrum.

We have used the values obtained for the moments $\{\overline{t}_p(E_0, E^0)\}^\Gamma$ and $\{\overline{t^2_p}(E_0, E^0)\}^\Gamma$, to calculate the distribution function for the number of electrons of energy greater than E^0, at a given depth t, in showers initiated by photons of energy E_0. We have approximated the function $\{N_p(E_0, E^0, t)\}^\Gamma$ by a sum of Laguerre polynomials $L^1_n(x)$:*

$$\{N_p(E_0, E^0, t)\}^\Gamma = (\gamma t)\, e^{-\gamma t} \sum_{n=0}^{K} A_n L^1_n(\gamma t),$$

$$L^1_0(x) = 1; \quad L^1_1(x) = 2 - x; \tag{2}$$

$$L^1_2(x) = 2 - 3x + x^2/2, \ldots$$

The orthogonality conditions on the polynomials yield the coefficients

$$A_n = \frac{\gamma}{n+1} \int_0^\infty N(E_0, E^0, t)\, L^1_n(\gamma t)\, dt. \tag{3}$$

These are simply related to the moments:

$$A_0 = \gamma K_\Gamma \frac{E_0}{\beta}; \quad A_1 = \frac{\gamma}{2} K_\Gamma \frac{E_0}{\beta} \{2 - \gamma \{\overline{t}_p(E_0, E^0)\}^\Gamma\};$$

$$A_2 = \frac{\gamma}{3} K_\Gamma \frac{E_0}{\beta} \{3 - 3\gamma \{\overline{t}_p(E_0, E^0)\}^\Gamma + \frac{\gamma^2}{2} \{\overline{t^2_p}(E_0, E^0)\}^\Gamma\}; \ldots$$

The coefficient γ is set equal to the minimum value of the total absorption coefficient of the most penetrating part of the radiation — the photons. $K_\Gamma(E_0, E^0)$ is defined by the expression

$$K_\Gamma(E_0, E^0) = -\frac{\varepsilon^0}{\varepsilon_0} e^{\varepsilon^0 - \varepsilon_0} + 1 + \varepsilon^0 e^{\varepsilon^0} (\mathrm{Ei}(-\varepsilon^0) - \mathrm{Ei}(-\varepsilon_0)) - \frac{\varepsilon^0}{\varepsilon_0^2} (1 - e^{-\varepsilon_0 + \varepsilon^0});$$

$$\varepsilon = \frac{Eq}{\beta}; \quad q = 2{,}29,$$

where β is the critical energy for a given substance.

The distribution function $\{N_p E_0, t, E^0)\}^\Gamma$ computed according to Eq. (2), is shown plotted in Fig. 1 to 3. The areas under the curve equal

$$\int_{E^0}^{E_0} P_\Gamma(E_0, E)\, dE = \frac{E_0}{\beta} K_\Gamma.$$

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* Approximation (2) is obtained by generalizing the approximation obtained in Ref. 3 for the function $N(E_0, 0, t)$.

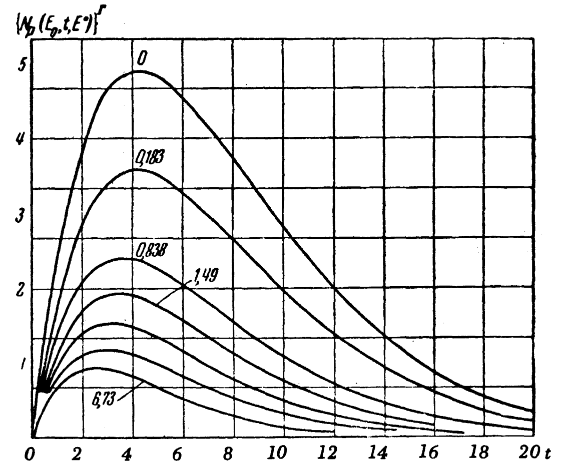

FIG. 1. Electron distribution functions in lead. The energy of the primary photon is $\varepsilon_0 = 118.1$. The values of ε_0 are indicated on the curves. The values of ε_0 for the second and the third curves from the bottom are respectively $\varepsilon_0 = 4.11$, and $\varepsilon^0 = 2.48$.

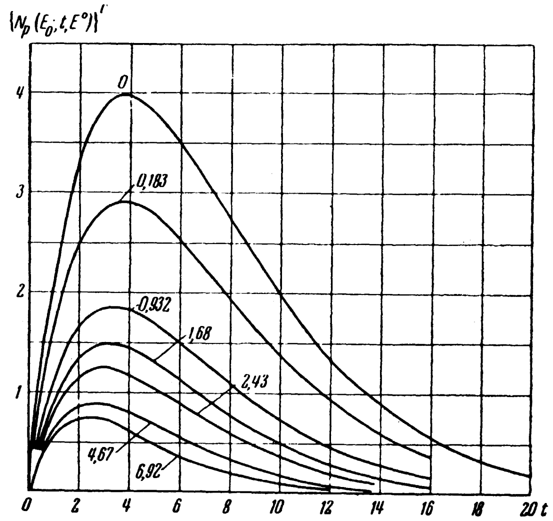

FIG. 2. Electron distribution functions in lead. The energy of the primary photon is $\varepsilon_0 = 90$. The values of ε^0 are indicated on the curves.

TABLE II.

$\varepsilon_0 = 50$		$\varepsilon_0 = 90$		$\varepsilon_0 = 118{,}1$	
ε^0	$K_\Gamma(\varepsilon_0, \varepsilon^0)$	ε^0	$K_\Gamma(\varepsilon_0, \varepsilon^0)$	ε^0	$K_\Gamma(\varepsilon_0, \varepsilon^0)$
0	1	0	1	0	1
0,023	0,924	0,023	0,924	0,023	0,924
0,072	0,836	0,072	0,836	0,072	0,836
0,107	0,790	0,107	0,790	0,107	0,790
0,183	0,715	0,183	0,715	0,183	0,715
0,738	0,413	0,932	0,417	0,838	0,438
1,292	0,354	1,680	0,307	1,493	0,329
2,402	0,246	2,428	0,245	2,476	0,242
4,511	0,190	4,674	0,155	4;114	0,171
5,175	0,142	6,919	0,114	6,734	0,117
6,839	0,113				

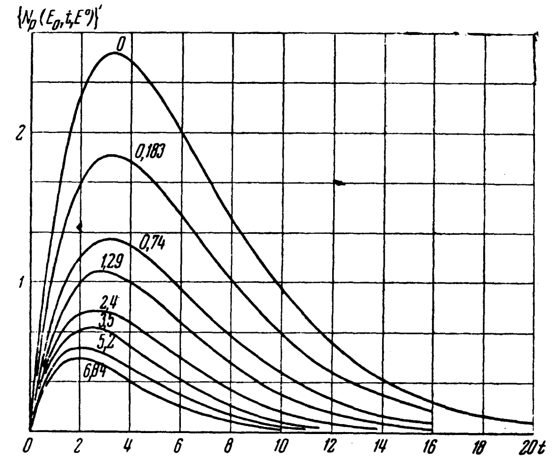

FIG. 3. Electron distribution functions in lead. The energy of the primary photon is $\varepsilon_0 = 50$. The values of ε^0 are indicated on the curves.

The values of $K_\Gamma(E_0, E^0)$ are listed in Table II for various values of E_0 and E^0. Note that for $E_0 \gg \beta$, the function $K_\Gamma(E_0, E^0)$ shows only slight dependence on E_0, and changes considerably as E^0 varies near zero. Fig. 4 shows the "equilibrium" spectrum obtained by Tamm and Belen'kii[9]; this figure also shows the energy spectra in the region of cascade maximum** constructed from the curves of Fig. 1 to 3. It may be seen in Fig. 4 that the energy spectra in the region of cascade maximum for $\epsilon_0 = 118.1$, 90, 50 agree with each other to within 2%, and agree within 10% with the "equilibrium" spectrum of Tamm and Belen'kii whose accuracy has been estimated several times.[10]

**As shown in cascade theory (see Ref. 13) the "equilibrium" energy spectrum resembles the spectrum found in the region of maximum spread of the cascade.

TABLE III.

t	1	2,6	4	6	8	10
$\varepsilon_0 = 118{,}1$	1,25	1,1	1,00	1,08	1,12	1,06
$\varepsilon_0 = 90$	1,11	1,00	1,01	0,98	1,08	1,22
$\varepsilon_0 = 50$	0,87	0,94	0,96	1,16	1,20	

TABLE IV.

	t	6	8	10	12
$\varepsilon_0 = 118{,}1$	S_{Pb}	1	1,1	1,2	1,3
	S_{light}	1,2—1,3	1,4—1,5	1,6—1,7	1,8—1,9
$\varepsilon_0 = 90$	S_{Pb}	1	1,1	1,2	1,3
	S_{light}	1,3—1,4	1,5—1,6	1,7—1,8	1,8—1,9

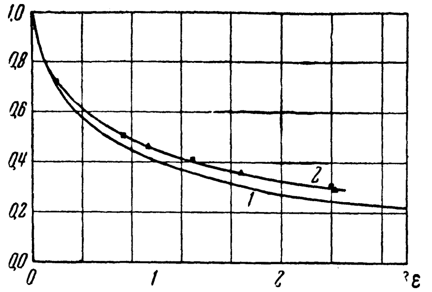

FIG. 4. Curve *1*: "equilibrium" energy spectrum; curve *2*: spectrum for maximum cascades initiated by primary photons of energy $\varepsilon_0 = 118.1$; the triangles give the spectrum for maximum cascades due to primary photons of energy $\varepsilon_0 = 90$, the squares, for maximum cascades due to primary photons of energy $\varepsilon_0 = 50$.

Let us assume that there are $N(E_0, 0, t_i)$ particles at a depth t_i; these particles have energy $E = \{1 - K(t_i)\}E_0$, where $K(t_i) = \frac{\beta}{E_0}\int_0^{t_i} N(E_0, 0, t)\,dt$ is determined by integrating $\{N_p(E_0, 0, t)\}^\Gamma$ (Fig. 1, 2, 3). We can therefore determine the average energy,

$$\overline{E}_{\text{area}} = \{1 - K(t_i)\}\, E_0 \,/\, N(E_0, t_i, 0),$$

of a particle at a depth t_i. On the other hand, we can use the curves $\{N_p(E_0, t, E^0)\}^\Gamma$ of Fig. 1 to 3 to determine the energy spectra of the particles and thus the average particle energy at a depth t_i.

Table III shows the value of the ratio of the average energies, $a = \overline{E}_{\text{spect}}/\overline{E}_{\text{area}}$, computed in these two ways; they agree within 20% for all values of the depth t_i. Obviously, damping is considerably faster in a light than in a heavy material at the cascade maximum. Therefore, at a given depth t_i, the energy spectrum of cascade particles will be softer in a light than in a heavy material at the cascade maximum. In order to verify whether this rule is fulfilled by the approximate curve *2*, we have evaluated the energy spectra for cascade particles according to Eq. (15.10) of Ref. 11 for various values of the cascade parameter.* We have then used these to determine the cascade parameter for the approximate curves *2*, Fig. 1 to 3, at a depth $t_i = 6, 8, 10, 12$ radiation units. We have also evaluated the cascade parameter s for a light material at the same depths t_i, by means of Eq. (15.15) of Ref. 11. The results of these calculations are presented in Table IV, from which it may be seen that for all depths, the cascade parameter s is larger in a light than in heavy material at the same depth. Therefore, in curves *2* the energy spectrum for cascade particles is softer in a light than in a heavy

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*In cascade theory the parameter s characterizes the "softness" of the energy spectrum of cascade particles: the larger is s, the softer is the energy spectrum.

material. We conclude from this that the "equilibrium" spectrum of Tamm and Belen'kii appears correct.

The "equilibrium" spectrum may be computed from the curves 2 appearing in Figs. 1 to 3, by integrating over the depth. It agrees to 1% with the expression obtained in Ref. 9, which permits us to judge the accuracy of the numerical computations. Thus the cascade curves $\{N_p E_0, t, E^0)\}^{\Gamma}$ obtained from Eq. (2) for lead, are accurate to 5 to 10%. We recall that the curves of Fig. 1 to 3 were obtained by including the energy dependence of the total absorption coefficient of photons and the Rutherford scattering of charged particles. The formula obtained by Fainberg[6] for the moments $\overline{t^n}(E_0,0)$ for an arbitrary spectrum of primary particles neglecting scattering, is easily generalized to include Rutherford scattering of charged particles for the moments $\overline{t^n}(E_0, E^0)$. Carrying out the corresponding calculation for the case when the photon spectrum in the boundary layer of the material has the form $\Gamma(E_0, 0, E, \theta) = \Phi_\gamma(E_0, E)\delta(\theta)$, where E_0 is the upper limit of the spectrum, we obtain the following formula:

$$\{\overline{t_p^n}(E_0, E^0)\}^{\Phi_\gamma} = \int_{E^0}^{E_0} \Phi_\gamma(E_0, E)\{\overline{t_p^n}(E, E^0)\}^{\Gamma} \times$$

$$\times \int_{E^0}^{E} P_{\Gamma,0,0}(E, E')\,dE'dE \Bigg/ \int_{E^0}^{E_0} P_{0,0}(E_0, E)\,dE;$$

$$\{\overline{t_\Gamma^n}(E_0, E^0)\}^{\Phi_\gamma} = \int_{E^0}^{E_0} \Phi_\gamma(E_0, E)\{\overline{t_\Gamma^n}(E, E^0)\}^{\Gamma} \times$$

$$\times \int_{E^0}^{E} \Gamma_{\Gamma,0,0}(E, E')\,dE'\,dE \Bigg/ \int_{E^0}^{E_0} \Gamma_{0,0}(E_0, E)\,dE. \qquad (4)$$

The quantities $P_{0,0}(E_0, E)$ and $\Gamma_{0,0}(E_0, E)$ are determined from the formulas

$$P_{0,0}(E_0, E) = \int_{E}^{E_0} \Phi_\gamma(E_0, E')\,P_{\Gamma,0,0}(E', E)\,dE';$$

$$\Gamma_{0,0}(E_0, E) = \int_{E}^{E_0} \Phi_\gamma(E_0, E')\,\Gamma_{\Gamma,0,0}(E', E)\,dE'.$$

The remaining quantities are computed from our Eq. (1).* The moments $\{t_p(E_0, E^0)\}^{\Phi_\gamma}$ and $\{\overline{t_p^2}(E_0, E^0)\}^{\Phi_\gamma}$ were computed for lead in Ref. 7, including Rutherford scattering of charged particles and energy dependence of the total absorption coefficient of photons, from a spectrum of primary photons, $\Phi_\gamma(E_0, E)$, having the form:

$$\Phi_\gamma(E_0, E) = \begin{cases} 1/E & \text{for } E < 330 \text{ Mev}, \\ 0 & \text{for } E > 330 \text{ Mev}. \end{cases} \qquad (5)$$

The results of these calculations were compared with the corresponding values computed from the experimental data of Blocker, Kenny and Panofsky,[12] assuming $E^0 = 0.5$ Mev; the computed and experimental values of $\{\overline{t_p^n}(E_0, E^0)\}^{\Phi_\gamma}$ agree to 4%. However, the moments $\overline{t_p^n}(E_0, E^0)$ are very sensitive functions of E^0 in the neighborhood of zero. The correctness of the choice $E^0 \approx 0.5$ Mev may be verified as follows: The authors of Ref. 12 have measured the areas under the cascade curves for the photon spectrum (5) in various materials. They have obtained particularly accurate values for the cases of Cu and Pb as the cascade curves for these elements were measured to a distance of 20 and 40 "cascade" units respectively. According to the data of Ref. 12:

$$\int_0^\infty N(E_0, t, E^0)\,dt = 0{,}946\,\frac{E_0}{\beta} \quad \text{in Cu and}$$

$$\int_0^\infty N(E_0, t, E^0)\,dt = 0{,}846\,\frac{E_0}{\beta} \quad \text{in Pb.}$$

Making use of our results listed in Table II, both of these values for the areas may be satisfied with E^0 - 0.2 Mev.**Note that the results which have been obtained, apply to experiments in which the measuring apparatus is surrounded by absorbers on all sides,[13] thus excluding edge effects which we have neglected.

The author wishes to express his gratitude to Professor S. Z. Belen'kii for valuable advice.

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*The formulas of Ref. 6 for the moments $\{\overline{t_p^n}(E_0, E)\}^{\Phi_\gamma}$, $\{\overline{t_p}(E_0, E^0)\}^{\Gamma}$ and $\{\overline{t_p^2}(E_0, E^0)\}^{\Gamma}$ are in error.

**Note that the resulting spectra are somewhat uncertain in the region of such small energies; besides, the experimental value of $\{\overline{t_p^n}(E_0, E^0)\}^{\Phi_\gamma}$ is 4% smaller than the theoretical value. Thus the actual value of E^0 lies between 0.2 and 0.5 Mev.

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Characteristic Frequencies and Amplitudes of Free Normal Oscillations in KCl Crystals

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Results are presented for a calculation of the characteristic frequencies and amplitudes of free normal oscillations of the ions of a KCl crystal. These calculations are carried out for values of the wave vector **k** which uniformly cover the cell of the reciprocal lattice by 729 points, with account taken of polarization deformation of the electron shells of the ions. A comparison is made with the results obtained by other authors.[1,2]

KNOWLEDGE of the characteristic frequencies and amplitudes of the free normal oscillations of the ions of a crystal is necessary in the solution of a number of problems, for example, in the determination of heat capacity, the interaction of a conduction electron with the vibrations of the lattice, the energy of interaction of point charges placed in a crystal, computation of the energy levels of the local states of an electron or hole with small radius of the state, etc.[3]

The frequencies $\omega_{\mathbf{k}}^{\alpha}$ for certain wave numbers **k** were found by Iona.[1] The amplitudes were not of interest since he computed the heat capacity. In determining the frequencies, Iona considered a model of a point lattice with ions of equal mass. Tolpygo and Zaslavskaia[2] computed the frequencies and amplitudes of the normal vibrations of a KCl crystal for 8 values of the vector **k** which, in particular, were obtained in Ref. 1 by considering the deformation of the electron shells of the ions, according to the method proposed earlier by Tolpygo.[4,5]

However, part of the exchange integral of the interaction of two (opposite) neighbors, which depends on the polarization, was thrown away in Ref. 2, without any justification. This can be avoided.[6] Moreover, knowledge of the frequencies for only 125 points of the reciprocal lattice is insufficient for the following quantitative analysis.

Let us consider an ideal crystal of KCl. We denote the dipole moment of the electron shell s_1 of the lth ion by $\mathbf{P}^{l}_{s}$, and the dipole moment, due to a shift $\mathbf{u}^{l}_{s}$ of this ion from its equilibrium position, by $\mathbf{p}^{l}_{s} = e_s \mathbf{u}^{l}_{s}$ (e_s = charge of the *s*th ion). For the