fissions from the galvanometer scale, λ is the disintegration constant, N is the number of particles bombarding the target during irradiation, f is the geometric factor, t_1 is the irradiation time of the target by the beam, t_2 is the time of onset of charge accumulation in the collecting electrode, and t_3 is the time of the cessation of accumulation. Before the yield was measured, the disintegration constant λ was determined, and the decay period of N^{13} was computed which, according to our data, proved to be 10.02 ± 0.1 min, in agreement with the findings of other authors.

Figures 3 and 4 show the yield curves for reactions (1) and (2). The ordinate indicates the yield, i.e., the number of positrons per particle bombarding the target, and the abscissa the energy of the bombarding particles in kilo-electron-volts.

From the yield curves one can determine the cross section by the familiar equation

$$\sigma = (dy/dE)(dE/dx)/n,$$

where dE/dx is the loss of energy by the bombarding particles in the target, y is the reaction yield and n is the number of nuclei per 1 cm^3. The quantities dy and dE are determined from the yield curve. Specific energy losses by the bombarding particles in the target are computed from the familiar Bethe formula.

Figure 5 shows the $C^{12}(p, \gamma)$ cross section curve. The value of the $C^{12}(p, \gamma)$ effective cross section is 0.30×10^{-30} cm^2 for an energy of 313 kev and grows to 6.4×10^{-30} cm^2 at 358 kev. The value of the $C^{12}(d, n)$ effective cross section is 0.8×10^{-28} cm^2 for an energy of 340 kev. The absolute errors in the determined cross sections do not exceed ±10%.

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Equilibrium Spectrum of Electrons and Photons with Account of Scattering

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An exact solution of the equation for the equilibrium spectrum of electrons and photons has been found with account of scattering, i.e., an expression is obtained for the angular and energetic distributions of particles at the shower maximum in heavy elements.

AN analytical expression for the "equilibrium" spectrum (integrated over the depth t *) of photons and electrons, which is valid for any dependence of the absorption coefficient of the photons $\sigma(E)$ on the energy, was first obtained in the work of Tamm and Belen'kii.[1] An analytical expression for the "equilibrium" spectrum, with account of the Rutherford scattering of the charged particles, was found by Belen'kii and Maksimov.[2] However, the equations in Ref. 2 for the equilibrium spectrum with account of scattering were solved approximately; making use of the method of adjoint equations,[3] it is possible to find their exact solution.

Let us write the basic equations of cascade theory with account of scattering:

$$\cos\vartheta \frac{\partial P}{\partial t} = L_1[P(E_0, t, E, \vartheta), \Gamma(E_0, t, E, \vartheta)] + \frac{E_k^2}{4E^2}\Delta_\vartheta P(E_0, t, E, \vartheta), \quad (1)$$

$$\cos\vartheta \frac{\partial \Gamma}{\partial t} = L_2[P(E_0, t, E, \vartheta), \Gamma(E_0, t, E, \vartheta)].$$

* The depth t is measured in atomic units.

Here P and Γ are the distribution functions of the electrons and photons at the depth t, energy E and angles ϑ in the shower, produced by primary particles of energy E_0; L_1 and L_2 are integro–differential operators which act on the variable E and take into account pair formation by photons, bremsstrahlung and ionization losses of the electrons; Δ_ϑ is the Laplace operator, $E_k = 21$ mev.

We apply the following boundary conditions:

$$P(E_0, 0, E, \vartheta) = \delta(E - E_0)\,\delta(\vartheta); \qquad (2)$$

$$\Gamma(E_0, 0, E, \vartheta) = 0.$$

Integrating Eq. (1) over t from 0 to ∞ with account of the boundary conditions (2), and excluding from the second equation the function $\Gamma_p(E_0, E, \vartheta)$, we get for the function $P_p(E_0, E, \vartheta)$ an equation of the following form:

$$-\cos\vartheta\,\delta(E - E_0)\,\delta(\vartheta) \qquad (3)$$

$$= \int_E^\infty P_p(E_0, E', \vartheta)\,\varphi(E', E)\,dE'$$

$$- \mu_e(E)\,P_p(E_0, E, \vartheta)$$

$$+ \beta\frac{\partial P(E_0, E, \vartheta)}{\partial E} + \frac{E_k^2}{4E^2}\Delta_\vartheta P_p(E_0, E, \vartheta);$$

$$\varphi(E', E) = 2\varphi'(E', E) + W_e(E', E' - E);$$

$$\mu_e(E) = \int_0^\infty W_e(E, E')\,dE';$$

$$\varphi'(E', E) = \int_E^{E'} \frac{W_p(E'', E')\,W_e(E'', E)}{\sigma(E'')}\,dE'',$$

W_p and W_e are respectively the probability of processes of pair creation and the probability of bremsstrahlung.

Let us investigate the solution of Eq. (3) in the form of a series of Legendre polynomials:

$$P_p(E_0, E, \vartheta) = \sum_{n=0}^{\infty} \varphi_n(E_0, E)\,P_n(\cos\vartheta). \qquad (4)$$

Substituting (4) in (3), multiplying (3) by $P_m(\cos\vartheta)$ and integrating over $\cos\vartheta$ from -1 to $+1$, we get the following equation for the function φ_n:

$$\left\{L + \beta\frac{\partial}{\partial E}\right\}\varphi_n(E_0, E) \qquad (5)$$

$$= -\frac{2n+1}{4\pi}\,\delta(E_0 - E),$$

where the operator L is given by the expression:

$$L\varphi_n(E_0, E) = \int_E^\infty \varphi_n(E_0, E')\,\varphi(E', E)\,dE' \qquad (6)$$

$$- \left\{\mu_e(E) + \frac{E_k^2}{4E^2}\,n(n+1)\right\}\varphi_n(E_0, E).$$

We determine the operator L^* so that

$$\int_{E_1}^\infty u_n(E, E_1)\,L\varphi_n(E_0, E)\,dE \qquad (7)$$

$$= \int_{E_1}^\infty \varphi_n(E_0, E)\,L^* u_n(E, E_1)\,dE,$$

where $u_n(E, E_1)$ is an arbitrary function satisfying the condition

$$u_n(E, E_1) = 0 \quad \text{for } E < E_1. \qquad (8)$$

The operator L^* operates on the variable E. Substituting (6) in (7) and changing the order of integration of E and E', we get

$$L^* u_n(E, E_1) = \int_{E_1}^E \varphi(E, E')\,u_n(E', E_1)\,dE'$$

$$- \left\{\mu_e(E) + \frac{E_k^2}{4E^2}\,n(n+1)\right\}u_n(E, E_1).$$

Multiplying Eq. (5) by $u_n(E, E_1)$ and integrating over E from 0 to ∞, we get:

$$\int_0^\infty u_n(E, E_1)\left[L + \beta\frac{\partial}{\partial E}\right]\varphi_n(E_0, E) \qquad (9)$$

$$= -\frac{2n+1}{4\pi}\,u_n(E_0, E_1).$$

Taking (8) into account, and also the fact that $\varphi_n(E_0, E) = 0$ for $E > E_0$, we have

$$\int_0^\infty u_n(E, E_1)\,\beta\,\frac{\partial}{\partial E}\,\varphi_n(E_0, E)\,dE \qquad (10)$$

$$= -\int_0^\infty \varphi_n(E_0, E)\,\beta\,\frac{\partial u_n(E, E_1)}{\partial E}\,dE.$$

Equation (10), together with (7) and (9), gives

$$\int_0^\infty \varphi_n(E_0, E)\left[L^* - \beta\frac{\partial}{\partial E}\right] u_n(E, E_1)\,dE \qquad (11)$$

$$= -\frac{2n+1}{4\pi}\,u_n(E_0, E_1).$$

If the function $u_n(E, E_1)$ in Eq. (11) is given, then Eq. (11) can be regarded as an integral equation for the function $\varphi_n(E_0, E)$. One can make use of the arbitrariness of the selection of the function $u_n(E, E_1)$ in that (11) was solved more easily than (5). We assume that the $u_n(E, E_1)$ satisfy the equation

$$[L^* - \beta\partial/\partial E]\,u_n(E, E_1) \qquad (12)$$

$$= -\delta(E - E_1)(2n+1)/4\pi.$$

It then follows from (11) that

$$u_n(E_0, E_1) = \varphi_n(E_0, E_1), \qquad (13)$$

i.e., solution of the primary equation (5) is equivalent to a solution of (11). We make use of (12) so as to obtain an equation for the determination of the energy part of the integral distribution function

$$N_p(E_0, E, \vartheta) = \int_E^\infty P_p(E_0, E, \vartheta)\,dE.$$

For this purpose, we integrate (12) over E_1 from E_2 to ∞, taking it into account that the operator L^* acts only on the variable E, and that when (12) is satisfied, $u_n(E, E_1) = \varphi_n(E, E_1)$. Substituting E for E_2 and E_0 for E, we obtain as a result

$$\left\{L^* - \beta\frac{\partial}{\partial E_0}\right\} f_n(E_0, E) = -\frac{2n+1}{4\pi}, \qquad (14)$$

$$f_n(E_0, E) = \int_E^\infty \varphi_n(E_0, E)\,dE.$$

Equation (12) is known as the associated equation to Eq. (5).

Let us consider the equation for the energy part of the nth component of the function N_p without account of the ionization loss:

$$L^* f_n(E, E_1) = -(2n+1)/4\pi. \qquad (15)$$

The solution of (15) was given by Belen'kii:[5]

$$u_n(E, E_1) = 0 \quad \text{for } E < E_1; \qquad (15')$$

$$u_n(E, E_1) = [(2n+1)/4\pi]\,qE^2/[(q^2E^2 + a_n^2\beta^2)(q^2E_1^2 + a_n^2\beta^2)]^{-1/2} \quad \text{for } E > E'$$

$$q = 2.29; \qquad a_n = {}^1/_2\,\varepsilon_k\sqrt{n(n+1)/q};$$

$$\varepsilon_k = E_k q/\beta.$$

This solution is exact for the value of $E \ll E_1$. The function $u_n(E, E_1)$ in Eq (11) depends on E; it satisfies the condition (8); in other respects, it is completely arbitrary. For u_n, we use the function (15′) and assume that it satisfies Eq. (15) exactly for arbitrary values of E.

The function $u_n(E, E_1)$ has a discontinuity for $E = E_1$. Consequently, for $E = E_1$, the following equation holds, (for $E^2q^2 > a_n^2\beta^2$):

$$\frac{\partial u_n(E, E_1)}{\partial E} = \frac{2n+1}{4\pi}\,\frac{1}{\sqrt{q^2E_1^2 + a_n^2\beta^2}} \qquad (16)$$

$$+ \frac{2n+1}{4\pi}\,\frac{E_1}{\sqrt{q^2E_1^2 + a_n^2\beta^2}}\,\delta(E - E_1).$$

Transforming to the function $f_n(E_0, E)$ and replacing E_1 by E, we get Eq. (11) in the following form:

$$f_n^p(E_0, E)\left\{1 + \beta\,\frac{1}{\sqrt{q^2E^2 + a_n^2\beta^2}}\right\} \qquad (17)$$

$$-\beta\,\frac{E}{\sqrt{q^2E^2 + a_n^2\beta^2}}\,\frac{\partial f_n^p(E_0, E)}{\partial E}$$

$$= \frac{2n+1}{4\pi}\,\frac{E_0}{\sqrt{q^2E^2 + a_n^2\beta^2}}.$$

For $a_n = 0$, it coincides with Eq. (77.4) of Ref.

3 for the "equilibrium" spectrum without account of scattering, obtained by the same method. Consequently,

$$f_0^p(E_0, E) = \frac{1}{4\pi}\frac{E_0}{\beta}\varepsilon e^{\varepsilon}\int_{\varepsilon}^{\varepsilon_0}\frac{e^{-x}}{x^2}dx. \qquad (18)$$

It is not difficult to find the solution of Eq. (17):

$$f_n^p(E_0, E) = \frac{2n+1}{4\pi}\frac{\varepsilon_0}{q} \qquad (18')$$

$$\times\exp\left(\sqrt{\varepsilon^2+a_n^2}\right)\varepsilon\left(\frac{\varepsilon}{a_n+\sqrt{\varepsilon^2+a_n^2}}\right)^{a_n}$$

$$\times\int_{\varepsilon}^{\varepsilon_0}\exp\left(-\sqrt{x^2+a_n^2}\right)\left(\frac{a_n+\sqrt{x^2+a_n^2}}{x}\right)^{a_n}\frac{dx}{x^2}.$$

For $a_n = 0$, Eq. (18′) coincides with (18). For $\epsilon >> a_n$, Eq. (18′) transforms to

$$f_n^p(E_0, E) = \frac{2n+1}{4\pi}\frac{\varepsilon_0}{q}e^{\varepsilon}\varepsilon\int_{\varepsilon}^{\varepsilon_0}\frac{e^{-x}}{x^2}dx$$

$$n = 0, 1, 2, \ldots$$

and under the condition $\epsilon_0 >> \epsilon$, we get an asymptotic expansion identical with the expansion of the equilibrium spectrum without account of scattering:

$$f_n^p(E_0, E) = \frac{2n+1}{4\pi}\frac{E_0}{q}\frac{1}{E}\left(1-\frac{2}{\varepsilon}\right)$$

$$n = 0, 1, 2, \ldots$$

For $\epsilon << a_n$, Eq. (18) becomes

$$f_n^p(E_0, E) = \frac{2n+1}{4\pi}\frac{\varepsilon_0}{q}\frac{1}{a_n+1},$$

$$n = 1, 2, 3, \ldots$$

while for f_0^p we obtain the following asymptotic expansion for small ϵ :[1]

$$f_0^p(E_0, E) = (E_0/4\pi\beta)(1+\varepsilon\ln\varepsilon+\varepsilon C+\ldots),$$

where $C = 0.5772$ is Euler's constant.

For an estimate of the approximation of the equation for the "equilibrium" spectrum of Belen'kii and Maksimov[2], we compute theratio $f_n^p(E_0 E)/f_n(\epsilon)$. The function f_n^p is determined by Eq. (18), while

TABLE

ε	0.0	0.2	0.5	1.0	2.0	5.0	10	25
$f_1^p(E_0, E)/f_1(\varepsilon)$	1.00	0.98	0.97	0.99	0.99	1.03	1.01	0.94
$f_2^p(E_0, E)/f_2(\varepsilon)$	1.00	1.00	1.01	1.03	0.99	1.00	1.02	1.06

$f_n(\epsilon)$ is the corresponding function computed in Ref. 2. The results of the calculations of this ratio are given in the Table. It is seen from the Table that the functions $f_n^p(E_0, E)$ and $f_n(\epsilon)$ coincide within the limits of 6%. We note that the expression for the "equilibrium" spectrum (18) is still approximate, since the initial equations are approximate (in them we made use of simplified expressions for the cross section of elementary processes, taking the Compton effect approximately into account). However, in Ref. 1, an approximate "equilibrium" spectrum (without account of scattering) was obtained from these equations; this spectrum did not differ from the exact value by more than 4.5%. The equations (1) take multiple scattering accurately into account, so that, evidently, the "equilibrium" spectrum (18) differs from the exact also by no more than 4.5%.

In the completion of this research, the author made use of the valuable advice of Prof. S. Z. Belen'kii.

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Behavior of an Electron in a Periodic Electric and a Uniform Magnetic Field

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The general form of the eigenfunction of an electron in a periodic electric and a uniform magnetic field is derived. The equation of motion and the quasi–classical energy levels are found for an electron with an arbitrary dispersion law in a magnetic field. The broadening of the discrete energy levels of an electron in a crystal in a magnetic field is calculated.

THE magnetic properties of a metal are determined on the basis of the magnetic properties of the electron "gas." The latter is closely connected with the energy spectra of the electrons. To elucidate the magnetic properties of metals, we can go by either of two paths. We can make an assumption on the concrete form of the dispersion law and on the basis of this assumption construct a theory, in the comparison of which with experiment several numerical parameters are determined. The approximations of weakly bound[1,2] or very strongly bound[2–4] electrons apply to such a type of assumption. The second path consists of a search for the connection of the magnetic properties of the electrons with the law of their dispersion in the general form. In this case the concrete form of the dispersion law which holds in each separate case can be determined from a comparison of theory with experiment (in particular, with experiments on the de Haas–van Alphen effect), although such a comparison is considerably more difficult than is shown above.

Such a course of action was first pointed out by I. Lifshitz and Kosevich,[5] who determined the energy levels of the electron and the magnetic susceptibility of an electron gas in the quasi-classical approximation.

In this research, a central assumption is that the Hamiltonian of an electron with an arbitrary dispersion law $E(p_x, p_y, p_z)$ in a magnetic field can be determined by replacing p_x, p_y, p_z by the components of the linear momentum operator $\hat{P}_x, \hat{P}_y, \hat{P}_z$.

The present paper is a continuation and development of the work reported in Ref. 5.

The first Section gives the general form of the exact eigenfunction of the electron in a uniform magnetic and a periodic electric field. Later, we give an approximate equation of motion of the electrons, which is shown to be identical to the Hamiltonian constructed by I. Lifshitz and Kosevich.[5] A quasi-classical solution of this equation and the energy levels of the electron have been found.

In the last Section, we consider the effect of broadening of the discrete energy levels of an electron (in a magnetic field) into narrow bands under the action of the periodic field of the lattice. The author has pointed out this broadening in previous papers.[1,2] In these researches, a calculation of the broadening of the levels was carried out in the approximations of weakly coupled and strongly coupled electrons. In the present work, this broadening is calculated outside the framework of the approximations pointed out; the results of the researches of Refs. 1 and 2 are entirely substantiated.

2. EIGENFUNCTIONS OF THE ELECTRONS

Let us write out the Schrödinger equation for an electron in a periodic electric potential V_p and a uniform magnetic field $H = H_z$ [the vector potential $A = (-Hy, 0, 0)$]:

$$\hat{H}\psi = -(\hbar^2/2m)\Delta\psi - (i\mu Hy\partial\psi/\partial x) + (e^2H^2y^2/2mc^2 + V_p)\psi = E\psi. \quad (1)$$

We introduce the translation operator of the electron $\hat{T}_{\mathbf{m}}$, $\mathbf{m} = (m_1a_1, 0, m_3a_3)$ is the vector of the lattice,