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Stability of Plasma in a Strong Magnetic Field

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The stability of an inhomogeneous plasma with respect to small perturbations in a strong magnetic field is investigated. The plasma density and temperature and also the magnetic field strength are considered as given functions of space coordinates and as parameters on which the plasma stability is dependent.

THE study of plasma stability is of interest to physicists and astrophysicists in connection with gaseous discharges. In the past few years, various types of plasma instability have been considered in a number of articles. Kruskal and Schwarzchild[1] have examined the instability of a plasma contained by a magnetic field in a gravitational field; they have also considered the instability of a plasmatic fiber resulting from small kinks. These authors describe the behavior of the plasma through hydrodynamical equations. This is only justified, however, when the collision frequency is large with respect to the frequency which characterizes the time rate of change of the disturbance. Brueckner and Watson[2] have considered some forms of instability arising when the plasmatic density is small and collisions may be neglected; the perturbed functions which describe the distribution of the plasma components were then assumed to satisfy a linearized Boltzmann equation with no collisions. Such equations are also used in the present article, as it is assumed that the processes which generate instability take place in a time considerably smaller than that required for a particle to travel through a mean free path.

It is further assumed that the initial disturbance arises in a volume which is small with respect to the dimensions of the system, and sufficiently far from the boundaries of the plasma. This allows us to neglect boundary conditions. Thus we limit ourselves to the consideration of a local instability which may be due to velocities and to inhomogeneities in density, temperature and magnetic fields.

1. THE DISTRIBUTION FUNCTION FOR THE STATIONARY STATE

The state of the plasma whose stability is to be examined is described through the distribution function of its components f_i (v, x), and by the electric and magnetic fields E and H. The index i denoting the plasma component (we assume the plasma consists of ions and electrons) will be left out whenever possible in order to simplify the notation. Assuming that E and H are constant in time, f_i is found to satisfy the equation

$$\mathbf{v}\partial f_i/\partial\mathbf{x} + (e_i\mathbf{E}/m_i + [\mathbf{v}\boldsymbol{\omega}_{Hi}])\,\partial f_i/\partial\mathbf{v} = \partial_e f_i/\partial t, \tag{1}$$

where m_i is the mass of the particles, e_i is their charge, and $\omega_{Hi} = e_i H/m_i c$ is the Larmor frequency. Equation (1) is coupled with Maxwell's equations by the well known expressions for the current and charge density

$$\mathbf{j} = \sum_i e_i n_i \mathbf{u}_i, \quad \rho = \sum_i e_i n_i, \qquad n_i\mathbf{u}_i = \int \mathbf{v} f_i d\mathbf{v}, \quad n_i = \int f_i d\mathbf{v}. \tag{2}$$

We shall consider later the case of a strong magnetic field, when the principal term in equation (1) becomes the term containing ω_H. This means that the Larmor radius $\lambda_L = \sqrt{T/m}\, mc/|e|H$ is much smaller than the dimensions of the system, the mean free path and the distance in which a particle receives an increment of velocity $v_T = \sqrt{T/m}$ from the action of the electric field E.

Assume[3] that the solution of Eq. (1) can be obtained in the form of an expansion in inverse power, of ω_H

$$f = f_0 + \omega_H^{-1} f_1 + \omega_H^{-2} f_2 + \cdots \tag{3}$$

The first two terms of the series are then found to obey the following equations

$$[\mathbf{v}\mathbf{l}]\frac{\partial f_0}{\partial \mathbf{v}} = 0,$$

$$\mathbf{v}\frac{\partial f_0}{\partial \mathbf{x}} + \frac{e}{m}\mathbf{E}\frac{\partial f_0}{\partial \mathbf{v}} + [\mathbf{v}\mathbf{l}]\frac{\partial f_1}{\partial \mathbf{v}} = \frac{\partial_e f_0}{\partial t}, \quad (4)$$

where $\mathbf{l} = \mathbf{H}/H$ is in the direction of the tangent to the magnetic lines of force. Let v_1 and v_2 be the components of the velocity along the coordinates perpendicular to $\mathbf{l}$, and let $\theta = \arctan(v_2/v_1)$; the first of equations (4) may then be written as $\partial f_0/\partial\theta = 0$. This implies

$$f_0 = F_0(\mathbf{x}, v_l, v_\perp), \quad v_l = \mathbf{v}\mathbf{l}, \quad v_\perp = \sqrt{v_1^2 + v_2^2}. \quad (5)$$

We can integrate the second of equations (4) over θ from 0 to 2π. Making use of the facts that f_1 is periodic in θ, while F_0 and $\partial_e F_0/\partial t$ are independent of θ, we obtain the equation

$$v_l(\mathbf{l}\partial F_0/\partial\mathbf{x}) + E_l \partial F_0/\partial v_l = \partial_e F_0/\partial t, \quad (6)$$

where E_l is the component of the electric field E along the direction $\mathbf{l}$. One must keep in mind that in general v_l and $v_\perp$ are themselves functions of x. In view of Eq. (6), f_1 may easily be obtained from Eq. (4). The function f_1 will contain the unknown functions F_1 which may be determined like F_0 from periodicity conditions in the next approximation.

In order not to complicate any further formulas which are already complex enough, we shall limit ourselves to investigating the stability of cylindrically symmetrical plasma. We introduce cylindrical coordinates r, φ, z, assuming that the function f depends only upon r. We assume further that the current flows along the z-axis, giving rise to a magnetic field H in the φ direction, and that $E_\varphi = 0$. Equation (6) then yields

$$F_0 = n(m/2\pi T)^{3/2} e^{-mv^2/2T}, \quad (7)$$

where n and T are the density and temperature, which are the same for the two plasma components (the temperature is given in energy units). The function f_1 has the form

$$f_1 = \left\{\frac{eE_r}{T} - \frac{1}{n}\frac{\partial n}{\partial r} - \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial r}\right\} v_\perp \sin\theta\, F_0 + F_1, \quad (8)$$

$$v_\perp = \sqrt{v_r^2 + v_z^2}, \quad \theta = \operatorname{arc\,tg}(v_z/v_r).$$

where F_1 is an undetermined function which we shall not need later. We also set $E_z = 0$ in the first approximation (7), since in the static case, the electromagnetic drift in the r-direction is compensated by diffusion which is only considered in the next approximation. Substituting (8) into (2) and applying Maxwell's equations, we find the following relation between the magnetic field H and the pressure $p = 2nT$:

$$\frac{1}{r}\frac{\partial rH}{\partial r} = -\frac{4\pi}{H}\frac{\partial p}{\partial r}. \quad (9)$$

Equations (7) to (9) will be required to investigate the stability of the system. The quantities H, n and T will be considered as given functions of r and will be treated as parameters on which the plasma stability depends.

2. THE FORM OF THE EQUATION FOR SMALL DISTURBANCES

We shall describe small disturbances δf_i, δ E, and δH in the stationary quantities f_i, E and H by a system of equations consisting of Maxwell's equations and the linearized Boltzmann equation

$$\frac{\partial\delta f_i}{\partial f} + \mathbf{v}\frac{\partial\delta f_i}{d\mathbf{x}} + \frac{e_i}{m_i}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}]\right)\frac{\partial\delta f}{\partial\mathbf{v}} \quad (10)$$

$$= -\frac{e_i}{m_i}\left(\delta\mathbf{E} + \frac{1}{c}[\mathbf{v}\delta\mathbf{H}]\right)\frac{\partial f_i}{\partial\mathbf{v}},$$

where collision terms are neglected.

Equation (10), which contains on its right-hand side the function f_i determined in the previous Section, is a homogeneous equation in the perturbations, with coefficients depending on r. As stated above, the initial disturbance, produced at a time $t = 0$, occupies a small volume with respect to the dimensions of the system. The expansion of small scale disturbances δa into Fourier series of a complete set of normalized functions, $\delta a = \Sigma\, a_m \psi_m\,(x)$, must include terms of large m. But then for large m, the functions ψ_m can be written as

$$\psi(\mathbf{x}) = \bar{\psi}(\mathbf{x})\, e^{iS(\mathbf{x})}, \tag{11}$$

where ψ and $\mathbf{k}(\mathbf{x}) = \partial S/\partial \mathbf{x}$ change only slightly over a distance $\lambda = 2\pi/k$. Accordingly we shall assume that the initial disturbance is of the form (11), and we shall seek solutions of the form

$$\delta f_i = g_i(\mathbf{xv})\, e^{i(\omega t + S)}, \tag{12}$$

$$\delta\mathbf{E} = \mathbf{e}(\mathbf{x})\, e^{i(\omega t+S)}, \qquad \delta\mathbf{H} = \mathbf{h}(\mathbf{x})\, e^{i(\omega t+S)}.$$

For simplicity, we assume harmonic time dependence $e^{i\omega t}$. It is, however, well known that such solutions of the kinetic equation with no collision integral lead to the appearance of divergent integrals. We shall therefore assume that the imaginary part of ω has a sign which insures the growth of the disturbance ($Im\omega < 0$). This specification corresponds to a Laplace transform. The resulting dispersion equation may be extended if needed in the upper half-plane, and determines the poles ω_k of the Laplace transforms of the functions δf, $\delta\mathrm{E}$ and $\delta\mathrm{H}$. For large values of t, the solution of Eq. (10) and Maxwell's equation has the form

$$A(t) \approx \sum A_k e^{i\omega_k t}, \tag{13}$$

where $A(t)$ is one of the functions representing the disturbance; therefore the existence of roots of the dispersion equation in the lower half-plane implies instability. Substituting (12) into 10) we obtain the following equation

$$i(\omega + \mathbf{kv})\, g + \mathbf{v}\frac{\partial g}{\partial \mathbf{x}} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\frac{\partial g}{\partial \mathbf{v}}$$

$$= -\frac{e}{m}\left(\mathbf{e} + \frac{1}{c}[\mathbf{vh}]\right)\frac{\partial f}{\partial \mathbf{v}}. \tag{14}$$

Since the size of the initial disturbance is much smaller than the dimensions of the system, the quantities g, h, e, n and T change only slightly over a distance $\sim \lambda$. The assumption of large magnetic field H implies that these quantities also change slightly over a distance equal to the Larmor radius λ. Terms containing $\partial g/\partial \mathbf{x}$ and the field E in Eq. (14) may therefore be assumed to be small, and Eq. (14) may be solved by the method of successive approximations.

In the zero approximation we have, instead of Eq. (14),

$$i(\omega + \mathbf{kv})\, g + [\mathbf{v}\omega_H]\frac{\partial g}{\partial \mathbf{v}} \tag{15}$$

$$= -\frac{e}{m}\left(\mathbf{e} + \frac{1}{c}[\mathbf{vh}]\right)\frac{\partial F_0}{\partial \mathbf{v}},$$

where F_0 is the Maxwell function (9). The second term of expansion (3) is included in the next approximation. If the quantity λ is smaller than the Larmor radius, which may happen for sufficiently weak magnetic field, then the magnitude of the velocity u will enter as a parameter in Eq. (15). N. N. Bogoliubov and S. L. Sobolev have shown that the plasma is then stable if u does not exceed a critical value u_{crit}. The condition $u < u_{crit}$ may also be fulfilled in a strong magnetic field. But in this case it does not necessarily follow that the plasma is stable as the Laplace transforms of the solutions of the basic equations have poles in the ω-plane – which are proportional to the first power of the gradients of the stationary quantities. In order to find these poles one must neglect the frequency ω in Equation (15) and take it into account in the next approximation.

Considering the above analysis, and assuming the geometry of the stationary state, we obtain from equation (15)

$$g = \frac{4\pi e\rho}{k^2 m v_\perp}\frac{\partial F_0}{\partial v_\perp} \tag{16}$$

$$+ G(r, v_\varphi v_\perp)\exp\left\{i\,\frac{v_\perp k}{\omega_H}\sin(\theta - \alpha)\right\},$$

where $\rho = \Sigma\, e_i \int g_i\, d\mathbf{v}$.

It is assumed here that the wave vector k is perpendicular to the magnetic field, i.e., the initial disturbance does not depend on the angle φ; k is then given by

$$k = \sqrt{k_r^2 + k_z^2}, \quad \alpha = \operatorname{arc\,tg}(k_z/k_r).$$

In that case

$$S(\mathbf{x}) = \int^r k_r(r)\, dr + k_z z,$$

where k_z is independent of the coordinates.

The function G is determined in the next approximation from the condition of periodicity with respect to θ and is found to be

$$G=\frac{4\pi}{k^2H}\frac{\left\{\frac{1}{n}\frac{\partial n}{\partial r}+\left(\frac{mv^2}{2T}-\frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial r}+\frac{\Omega}{k_z}\frac{eH}{cT}\right\}\left\{c\rho J_0(w)-\frac{v_\perp}{c}jJ_0'(w)\right\}}{\frac{\Omega}{k_z}-\frac{v_\perp^2}{2}\frac{\partial}{\partial r}\left(\frac{1}{\omega_H}\right)-\frac{v_\varphi^2}{r\omega_H}}F_0,$$

$$\Omega=\omega-\frac{cE_r}{H}k_z,\qquad j=\frac{k}{ik_z}\sum e_i\int v_\perp\cos\theta g_i\,dv,$$

J_0 (ω) and J_0' (ω) are the zero order Bessel function and its derivative ; $\omega = v_\perp k/\omega_H$

Substituting (16) into (12) and eliminating the unknown quantities j and ρ, we obtain the dispersion equation . If it is assumed that the wave length λ is larger than the Debye and Larmor radii

$$(k_0^2 = k^2T/4\pi e^2n \ll 1$$

and

$$k_L^2 = k^2Tmc^2/e^2H^2 \ll 1),$$

the dispersion equation becomes

$$\left(2-\sum_i Q_i^{(0)}\right)\left(4\eta+\sum_i Q_i^{(2)}\right) \tag{17}$$
$$+\left(\sum_i \varepsilon_i Q_i^{(1)}\right)^2+\left(k_0^2+\sum_i k_L^2 Q_i^{(1)}\right)\left(4\eta+\sum_i Q_i^{(2)}\right)$$
$$-\frac{1}{2}\sum_i k_L^2 Q_i^{(3)}\left(2-\sum_i Q_i^{(0)}\right)$$
$$-\frac{3}{2}\sum_i \varepsilon_i k_L^2 Q_i^{(2)}\sum \varepsilon_i Q_i^{(1)}=0,$$

where

$$\eta=H^2/8\pi p,\ p=2\pi T,\ \varepsilon_i=\pm 1$$

for ions and electrons respectively, and

$$Q_i^{(n)}=\frac{1}{\sqrt{\pi}}\int_0^\infty \xi^n e^{-\xi}d\xi\int_{-\infty}^{\infty}e^{-\tau^2} \tag{18}$$
$$\times\frac{\frac{\Omega}{k_z}\frac{e_iH}{cT}+\frac{\partial\ln nT^{-3/2}}{\partial r}+(\xi+\tau^2)\frac{\partial\ln T}{\partial r}}{\frac{\Omega}{k_z}\frac{e_iH}{cT}+\frac{\partial\ln H}{\partial r}\xi-\frac{2}{r}\tau^2}d\tau.$$

Formula (18) determines $Q_i^{(n)}$ in the lower half of the complex ω-plane. If $Im\omega \geq 0$, the contour of integration must be suitably deformed.

3. THE DISPERSION EQUATION

A. *Small Curvature of the Magnetic Field*

Let $\partial H/\partial r \gg H/r$. The last term in the denominator of (18) can then be neglected. Equation (17) then splits up into two equations. One of these is a second degree algebraic equation which admits the solution

$$z\equiv\frac{H^2}{2\pi p}\frac{\Omega}{k_zv_{dr}}=\left(\frac{1}{2}-\frac{H^2}{8\pi p}\right) \tag{19}$$
$$\pm\sqrt{\left(\frac{1}{2}+\frac{H^2}{8\pi p}\right)^2-\frac{\partial\ln T}{\partial\ln H}},$$
$$v_{dr}=\frac{c}{enH}\frac{\partial p}{\partial r},$$

where v_{dr} is the drift velocity of the electrons with respect to the ions. The second equation has the following form:[5]

$$F(z)\equiv-2+(z+\gamma)\Phi(z)$$
$$+(-z+\gamma)\Phi(-z)=0, \tag{20}$$

$$\Phi(z)=\int_0^\infty\frac{e^{-\xi}}{z+\xi}d\xi,\qquad \gamma=\frac{\partial\ln nT^{-1}/\partial\ln H}{1-\partial\ln T/\partial\ln H}, \tag{21}$$

where γ is a parameter on which depends the plasma stability.

One of the stability conditions turns out to be that the radicand in equation (19) be positive. Since the function $F(z)$ has no singularities below the real axis, the number of roots N of equation(20) in the lower half-plane is equal to

$$N=\frac{1}{2\pi i}\int_C\frac{dF}{dz}\frac{dz}{F}. \tag{22}$$

The contour of integration, C, is shown in Fig. 1.

For large z, the function $F(z)$ takes the form

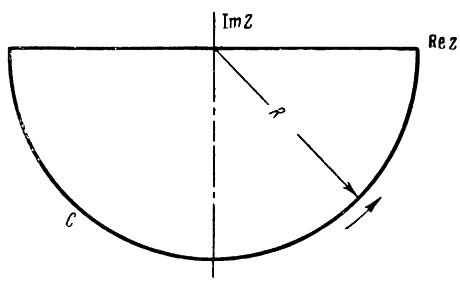

FIG. 1

$$F(z) = \frac{2(2-\gamma)}{z^2} + \frac{12(4-\gamma)}{z^4} + O\left(\frac{1}{z^6}\right). \quad (23)$$

The value of the integral (22) over the infinite semi-circular path is then $-2\pi i$. Since $F(z)$ is an even function of z, the integral along the real axis equals $2i$ times the increment in the argument of $F(z)$ as z goes from infinity to zero. Therefore

$$N = -1 - [\arg F(\infty) - \arg F(0)]/\pi. \quad (24)$$

For real values of z, the imaginary part of $F(z)$ differs from 0,

$$\operatorname{Im} F(z) = i\pi e^{-|z|}(|z| - \gamma)\operatorname{sign} z, \quad (25)$$

since the contour of integration for (21) must be deformed when the real axis is approached from the lower half-plane. If γ is positive, Im $F(z)$ vanishes for $z = \pm\gamma$ and Re $F(\pm\gamma) < 0$. Therefore equation (20) has no real roots. For small values of z, $F(z)$ may be taken as

$$F(z) = -2\gamma \ln z - 2(1 + C\gamma) - i\pi\gamma, \quad (26)$$

where $C \approx 0.577$ is Euler's constant.

The increase in the argument of $F(z)$ as z moves along the real axis can easily be obtained from Eqs. (23), (25), and (26). Thus when

$$0 < \gamma < 2 \quad (27)$$

the function $F(z)$ transforms the real axis into the contour shown in Fig. 2, in accordance with the formula obtained for it. It follows from Fig. 2 that the increase in the argument of $F(z)$ equals -2π and the number of roots N is then equal to 1. Therefore when condition (27) is satisfied, the plasma is unstable.

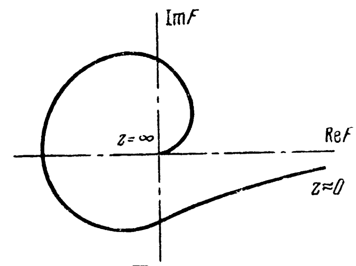

FIG. 2

One may verify in a similar fashion that for other values of γ, the number of roots N is zero, i.e., the plasma is stable. If instead of γ we introduce the parameter $\partial \ln T/\partial \ln H$, then in view of equation (19) we obtain the following condition for the plasma stability in a magnetic field of small curvature:

$$-\frac{H^2}{8\pi p} < \frac{\partial \ln T}{\partial \ln H} < \left(\frac{1}{2} + \frac{H^2}{8\pi p}\right)^2. \quad (28)$$

Condition (28) resembles the condition for convectional stability of an inhomogeneously heated gas in a gravitational field.[6] Note that the component $k_r(r)$ of the wave vector does not enter in Eqs. (19) and (20). This shows that when the wave length exceeds the Larmor and Debye radii, the solution is independent of the form of the initial disturbance along the radius r. When γ is close to 2 or 0, the expressions (23) and (26) approach zero, and the roots of Eq. (20) can be obtained in the explicit form

$$z = -i\sqrt{\frac{12}{2-\gamma}}, \quad \gamma \approx 2, \quad (29)$$

$$z = -ie^{-(1/\gamma)-C}, \quad \gamma \approx 0.$$

As expected, the real part of z is small in the first case and large in the second.

B. Intermediate Curvature of the Magnetic Field

Let us now assume that the current density and plasma temperature are constant over a cylindrical cross section. Then (see Ref. 7)

$$H = H_0 q,$$
$$n = (H_0^2/8\pi T)(1 - q^2) \quad \text{for} \quad q = r/a < 1, \quad (30)$$

where a is the radius of the cylinder which contains the plasma, $H_0 = 2I/ca$ and I is the total current. The dispersion equation then takes the form

$$F(z) \equiv \{2 + (\gamma - z)\,\Phi^{(0)}(z) + (\gamma + z)\,\Phi^{(0)}(-z)\} \times \{\gamma - (\gamma - z)\,\Phi^{(2)}(z) - (\gamma + z)\,\Phi^{(2)}(-z)\} + \{(\gamma - z)\,\Phi^{(1)}(z) - (\gamma + z)\,\Phi^{(1)}(-z)\} = 0, \tag{31}$$

$$\Phi^{(n)}(\pm z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \xi^n e^{-\xi} d\xi \int_{-\infty}^{+\infty} \frac{e^{-\tau^2} d\tau}{\pm z + \xi - 2\tau^2},$$

$$n = 0,\ 1,\ 2,$$

$$\gamma = 2q^2/(1 - q^2),$$

$$z = (\Omega\,|e|\,2I/k_z c^2 T)\, q^2, \quad q = r/a.$$

For large values of z,

$$F(z) = 4\,\frac{(\gamma - 2)(\gamma + 5/2)}{z^2} + 4\,\frac{13\gamma^2 - 25\gamma/2 - 198}{z^4} + O\left(\frac{1}{z^6}\right). \tag{32}$$

As in the previous case, the imaginary part of $F(z)$ differs from zero along the real axis. For large real values of z

$$\operatorname{Im} F(z) = \frac{\sqrt{2\pi}}{3}\, e^{-|z|/2} \left\{-(\gamma + 4)\sqrt{|z|} + \frac{\gamma^2 + 5\gamma/2 - 4/3}{\sqrt{|z|}} + O\left(\frac{1}{|z|^{3/2}}\right)\right\} \operatorname{sign} z. \tag{33}$$

For small values of z, the function $F(z)$ takes the form

$$F(z) = -1.61\,(1 + 0.76\gamma) - \frac{4\pi^2}{27}\gamma^2 + 1.61\sqrt{\pi}\,(1 + i)\,\gamma^2 \left(\frac{z}{2}\right)^{1/2} + \frac{4\pi i}{9\sqrt{3}}\,(6 + 11.08\gamma - 5.08\gamma^2)\,\frac{z}{2} + O(z^{3/2}). \tag{34}$$

When γ is close to 2, the roots of Eq. (31) may be obtained in explicit form by making use of expansion (32). Setting (32) equal to zero, we find

$$z = -i\sqrt{38/(2 - \gamma)} \qquad \gamma \approx 2. \tag{35}$$

It follows from Eq. (35) that the plasma is unstable for values of γ to the left of the point $\gamma = 2$, while it is stable for values of ν to the right of that point. We shall show that the plasma is unstable over the whole interval $\gamma = 0$ to $\gamma = 2$, or, what amounts to the same thing, for all values of $q = r/a$ which fulfill the condition

$$0 < r/a < 1/\sqrt{2}. \tag{36}$$

Indeed when γ is close to 2, the number of roots N may be found from equation (35). When $\gamma < 0$, equation (24) and equations (32) to (35) may be used to find the number of roots N, and it may be easily verified that the real axis is transformed by the function $F(z)$ into the contour shown in Fig. 3. Fig. 4 shows this contour for $\gamma > 2$, when the number of roots N in the lower half-plane is zero. We now vary continuously the value of γ from 2 to 0 and from 2 to ∞. If the contour intersects the origin during this variation, the number of turns changes by $\pm 2\pi m$ (where m is an integer) and the number of roots changes by $\pm 2m$. But in region (36) the number of roots can only increase, for otherwise the number N would be negative. Therefore the plasma is unstable over the whole interval (36).

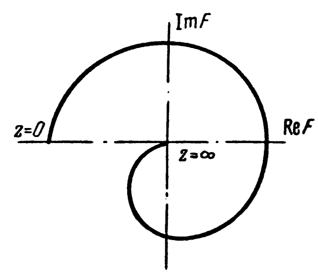

FIG. 3

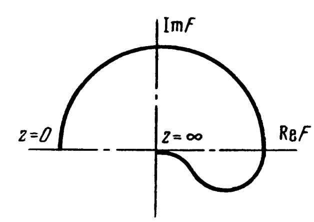

FIG. 4

If the system of equations,

$$\operatorname{Im} F(zq) = 0, \quad \operatorname{Re} F(zq) = 0 \quad (\operatorname{Im} z = 0), \tag{37}$$

which determines the real roots of Eq. (31) admits of solutions for values of the parameter q lying outside of the interval (36), then there may appear regions of instability in the interval $1/\sqrt{2} < q < 1$.

If $r < a/\sqrt{2}$, then instability may generally cause a redistribution of currents over the cross-section of the cylinder.

The dependence of n, T and H upon r may similarly be obtained inother cases.

In conclusion the author wishes to express his profound gratitude to N. N. Bogoliubov for his assistance in this analysis.

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Quantum Corrections to the Thomas–Fermi Equation

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The operator form of the Hartree–Fock equation is considered. The Thomas–Fermi and the Thomas–Fermi–Dirac equations are obtained in the zeroth approximation in terms of $\hbar$. Quantum corrections were found by the operator method for the Thomas–Fermi equations of 2nd and 4th order in $\hbar$. The correction of 2nd order is compared with the Weizsacker correction and it is shown that the latter is 9 times larger than the quantum theory value. The resultant equations are applied to the computation of the total energy of the atom.

1. INTRODUCTION

THE Thomas–Fermi method[1] is one of the methods of the statistical description of systems consisting of a large number of identical particles, and finds wide application in different areas of physics. On the basis of this method, the idea is presented of electrons (if an atom is under discussion) moving classically but with the additional condition that in each cell of phase space there be located no more than two particles. Interaction of particles is considered here by the introduction of the self–consistent field (with or without exchange).

The method under consideration is approximate, for which reason attempts have repeatedly been made at making it more precise in various ways by the introduction of corresponding corrections.[1] In their number we include the quantum correction (or, what amounts to the same thing, the correction for heterogeneity) which reflects the fact of the smearing out of the trajectory of the particle. This correction was first found by Weizsacker[2] by a variational method; however, in its quantitative behavior, it has been subjected to criticism, both in principle and in a comparison of its value with experiment.[3–7]. It was established that the Weizsacker correction was too large a quantity, in which connection, it was improved in a series of researches[3,6] by the introduction of a constant coefficient less than unity. In the present work, a stepwise quantum–mechanical derivation of the quantum corrections of second and fourth order in $\hbar$ is deduced from the Hartree–Fock equation. In this case it is appropriate to use the operator formulation of the problem. A study of the non–relativistic equation of Hartree–Fock in operator form is given in Sec. 2. This form is especially convenient in the relativistic case, and also for interactions which depend on the spin or on the isotopic spin. In the neglect of the non–commutability of the operators for the potential and kinetic energies, we obtain the Thomas–Fermi and the Thomas–Fermi–Dirac equations.

The quantum corrections correspond to a consideration of the commutators of these operators,

Other Errata

Page	Column	Line	Reads	Should Read
			Volume 4	
38	1	Eq. (3)	$\dots\frac{\pi r^2\rho^2\rho^2{}_n}{\rho_s^2},$	$\dots\frac{\pi r^2\rho^2\rho_n}{\rho_s^2},$
196		Date of submittal	May 7, 1956	May 7, 1955
377	1	Caption for Fig. 1	$\delta_{35}=\eta-21\cdot\eta^5$	$\delta_{35}=-21^\circ\eta^5\cdot$
377	2	Caption for Fig. 2	$\alpha_3=6.3^\circ\ \eta$	$\alpha_3=-6.3^\circ\eta$
516	1	Eq. (29)	$s^2/c^2\dots$	s/c
516	2	Eqs. (31) and (32)	Replace A_1s^2/c^2 by A_1	
497		Date of submittal	July 26, 1956	July 26, 1955
900	1	Eq. (7)	$\dots\frac{i}{4\pi}\sum_{c,\,\alpha}\frac{\partial w_a(t,\,P)}{\partial P^\alpha}\dots$	$\dots 2\pi^2 i\sum_{c,\,\alpha}\dots$
			(This causes a corresponding change in the numerical coefficients in the expressions that result from the calculation of the effects of the plasma particles on each other).	
804	2	Eq. (1)	$\dots\exp\{-(\overline{T}-V')\}$	$\dots\exp\{-(\overline{T}-V')\tau^{-1}\}$
			Volume 5	
59	1	Eq. (6)	$v_l(l\partial F_0/\partial x)+\dots$ where E_l is the projection of the electric field E on the direction 1	$\overline{(v\partial F_0/\partial x)}+\dots$ where the bar indicates averaging over the angle θ and E_l is the projection of the electric field E along the direction 1
91	2	Eq. (26)	$\Lambda=0.84\,(1+22/A)$	$\Lambda=0.84/(1+22/A)$
253		First line of summary	$Tl^{204,\,206}$	$Tl^{203,\,205}$
318	1	Figure caption	$\dots e^2mc^2=2.8\cdot10^{-23}$ cm,	$\dots e^2/mc^2=2.8\cdot10^{-13}$ cm,
398		Figure caption	... to a cubic relation. A series of points etc.	... to a cubic relation, and in the region 10 – 20°K to a quadratic relation. A series of points ●, coinciding with points O, have been omitted in the region above 10°K.