Determination of the Green's Function in the Bloch-Nordsieck Model by Functional Integration

A. V. SVIDZINSKII
Moscow State University
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In an electrodynamic model where the Dirac γ matrices are replaced by c-numbers, there is no vacuum polarization and the equations for the Green's function of an electron in an external field can be solved by quadratures. The Green's function with radiative corrections is then easily obtained by functional integration. The expression obtained for the Green's function after renormalization is manifestly analytic and has no infinities.

1. INTRODUCTION

SEVERAL papers[1-8] have recently appeared on functional integration as applied to quantum field theory. This method is of interest from two points of view: first, the mathematical apparatus seems the most adequate way of describing a field as a system with an infinite number of degrees of freedom, so that within the mathematical scheme one can obtain solutions of the field equations in closed form: while secondly the functional integration method does not assume the coupling constant to be small, so that one may make other approximations than those founded on perturbation theory.

However, although it has been possible to express the fundamental quantities in the theory as functional integrals, the method has not been able to produce concrete physical results because so far only the gaussian functional could be integrated.

From a methodological point of view, it is interesting at this stage to consider several simple models where the integration can be carried out. In Refs. 1 and 8, functional integration methods were applied to the interaction between an infinitely heavy nucleon and a scalar meson field, Ref. 1 considering a neutral and Ref.8 a charged field.

In the present paper, we consider, as an example where the calculations can be carried out in full, the the Bloch-Nordsieck model in electrodynamics. As is well known, Bloch and Nordsieck proposed to avoid the infrared catastrophe in the interaction of particles with light at low frequencies by an approximate method not based on perturbation theory. It turned out that their zero order approximation was equivalent to replacing the Dirac γ matrices by c-numbers u_α

$$u^2 = g^{\alpha\alpha}u_\alpha^2 = u_0^2 - \mathbf{u}^2 = 1. \qquad (1)$$

Here

$$g^{\alpha\alpha} = \begin{cases} 1 & \text{for } \alpha = 0, \\ -1 & \text{for } \alpha = 1, 2, 3. \end{cases} \qquad (2)$$

We shall look at this model from the Green's function point of view, and start with the representation, obtained in Ref. 3, for the Green's function of an electron as a functional integral depending on the Green's function of an electron in a classical external field. The integral will be calculated and then renormalized in the following sections.

First note that there is no vacuum polarization in this model, so that we may immediately assume

$$\langle S\rangle_{F_0} = \exp\left\{\sqrt{4\pi}\,e\int_0^1 d\lambda \int dx G(x, x\,|\,\lambda A)\,uA(x)\right\} = 1. \qquad (3)$$

Formula (3) is the well-known expression for the average of the S-matrix in a Fermi vacuum (see, for example Ref. 3); we use the Feynman notation for the scalar product of four-vectors:

$$ab = g^{\alpha\alpha}a_\alpha b_\alpha = a_0 b_0 - \mathbf{a}\cdot\mathbf{b}.$$

That there is no vacuum polarization is clear from the following considerations. The Green's function $S^c(x-x')$ for a free electron is determined in the present case by the first order equation

$$[iu_\alpha(\partial/\partial x_\alpha) - m]\,S^c(x-x') = -\delta(x-x') \qquad (4)$$

and has one pole, not two as in ordinary electrodynamics:

$$S^c(x-x') = \frac{1}{(2\pi)^4}\int \frac{e^{-ip(x-x')}}{m - up - i\varepsilon}\,dp. \qquad (5)$$

Hence $S^c(x-x')=0$ for $t'>t$, i.e., this is a retarded Green's function. Then, in the matrix element corresponding to a diagram of the type

one of the electron lines will correspond to $S^c \times (x-x')=0$. Similarly, all more complicated diagrams corresponding to vacuum polarization will be zero. Physically, the absence of a second pole in the Green's function means that there are no antiparticles in the theory, and hence that pairs cannot be created. Since the contribution of closed loops is zero, the photon Green's function in this model is identical with its zero order approximation in the perturbation theory.

2. GREEN'S FUNCTION FOR AN ELECTRON IN AN EXTERNAL FIELD

In the present case, the Green's function for an electron in a given external field is determined by the equation

$$[iu_\alpha(\partial/\partial x_\alpha) - \sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) - m] \times G(x, x' \mid A) = -\delta(x-x'), \tag{6}$$

which can be solved by quadratures. To do this, we use Fock's proper time method [10], which involves the formula

$$H^{-1} = -i\int_0^\infty d\nu e^{iH\nu-\varepsilon\nu}. \tag{7}$$

For our special case

$$H = iu_\alpha(\partial/\partial x_\alpha) - \sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) - m, \tag{8}$$

so that the solution of (8) can be written symbolically

$$G(x, x' \mid A) = i\int_0^\infty d\nu \exp\left\{i\nu\left(iu_\alpha\frac{\partial}{\partial x_\alpha} - \sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) - m + i\varepsilon\right)\right\}\delta(x-x'). \tag{9}$$

Introducing a new unknown function

$$U(\nu) = \exp\left\{i\nu\left(iu_\alpha\frac{\partial}{\partial x_\alpha} - \sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) - m + i\varepsilon\right)\right\}\delta(x-x'), \tag{10}$$

we find that it satisfies the differential equation

$$-i\frac{\partial U(\nu)}{\partial\nu} = \left(iu_\alpha\frac{\partial}{\partial x_\alpha} - \sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) - m + i\varepsilon\right)U(\nu) \tag{11}$$

with the initial condition with respect to the proper time parameter ν

$$U(0) = \delta(x-x'). \tag{12}$$

Taking the Fourier transform of the δ function

$$\delta(x-x') = (2\pi)^{-4}\int e^{-ip(x-x')}\,dp \tag{13}$$

we seek a solution of (11) in the form

$$U(\nu) = (2\pi)^{-4}\int \exp\{S(p, \nu)\}\,dp. \tag{14}$$

The function $S(p, \nu)$ satisfies the equation

$$-i\frac{\partial S}{\partial\nu} = iu_\alpha\frac{\partial S}{\partial x_\alpha} - \sqrt{4\pi}\,eg^{\alpha\alpha}u_\alpha A_\alpha(x) - m + i\varepsilon \tag{15}$$

with the initial condition

$$S(\nu; p)|_{\nu=0} = -ip(x-x'). \tag{16}$$

Finally, the substitution

$$S = -ip(x-x') - i(m - up - i\varepsilon)\nu + R, \tag{17}$$

leads to the following equation for R

$$-i\frac{\partial R}{\partial\nu} = iu_\alpha\frac{\partial R}{\partial x_\alpha} - \sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) \tag{18}$$

and the initial condition

$$R(0) = 0. \tag{19}$$

Equation (18), with the initial condition (19), is easily solved by Fourier transforms. The result is

$$R = -i\frac{\sqrt{4\pi}e}{(2\pi)^4}\int dku A(k)\, e^{-ikx}\int_0^{\nu} d\nu' e^{iuk\nu'}, \quad (20)$$

$$G(x, x' \mid A) = i\int_0^{\infty} d\nu \quad (21)$$

$$\exp\{-ip(x-x') - i(m-up-i\varepsilon)\nu + R(\nu \mid A)\}.$$

3. THE ELECTRON GREEN'S FUNCTION WITH RADIATIVE CORRECTIONS

As was shown in Ref. 3, the Green's function with radiative corrections can be written as the ratio of two functional integrals:

$$G(x-x') \quad (22)$$

$$= \frac{\int G(x, x' \mid A)\exp\{i\int L_f^0 dx\} < S >_{F_0} \delta A}{\int \exp\{i\int L_f^0 dx\} < S >_{F_0} \delta A}.$$

Here $\int L_0^f\, dx$ is the action function for an electromagnetic field and can be written

$$\int L_0^f dx \quad (23)$$

$$= \frac{g^{\alpha\alpha}}{2(2\pi)^4}\int A_\alpha(k)\, D^{c^{-1}}(k, k')\, A_\alpha(k')\, dkdk',$$

where

$$D^c(k, k') = D^c(k)\,\delta(k+k'),$$
$$D^c(k) = -1/(k^2 + i\varepsilon). \quad (24)$$

In our case we have the important simpification $< S >_{F_0} = 1$.

Hence we obtain

$$G(x-x') \quad (25)$$

$$= c^{-1}\int G(x, x' \mid A)\exp\left\{i\int L_f^0 dx\right\}\delta A,$$

where the denominator of formula (22), which is a constant, has been denoted by c. From the form of $G(x, x' \mid A)$ it is clear that in order to calculate (25) it is sufficient to find

$$\int \exp\left\{R(\nu \mid A)\right\} e^{i\int L_f^0 dx}\delta A \quad (26)$$

$$= \int \exp\left\{-\frac{i}{(2\pi)^4}\int g^{\alpha\alpha}F_\alpha(k)\, A_\alpha(k)\, dk\right.$$

$$\left. + \frac{i}{2(2\pi)^4}\int g^{\alpha\alpha}A_\alpha(k)\, D^{c^{-1}}(k, k')\, A_\alpha(k')\, dk\, dk'\right\}\delta A,$$

where for convenience we have written

$$F_\alpha(k) = \sqrt{4\pi}\, e u_\alpha e^{-ikx}\int_0^{\nu} d\nu' e^{iuk\nu'}; \quad (27)$$

The functional integral (25) is of the gaussian type and can easily be calculated with the help of the transformation

$$A_\alpha(k) = A'_\alpha(k) + \int F_\alpha(k')D^c(k, k')\, dk'. \quad (28)$$

With this substitution the integral takes the form

$$\int \exp\left\{R(\nu \mid A)\right\}\exp\left\{i\int L_f^0 dx\right\}\delta A \quad (29)$$

$$= c\exp\left\{-\frac{i}{2(2\pi)^4}\right.$$

$$\left.\int g^{\alpha\alpha}F_\alpha(k)\, D^c(k, k')\, F_\alpha(k')\, dkdk'\right\}.$$

Then, using (25) and (27), we obtain

$$\int \exp\{R(\nu \mid A)\}\exp\left\{i\int L_f^0 dx\right\}\delta A \quad (30)$$

$$= c\exp\left\{-\frac{ie^2}{2(2\pi)^3}\right.$$

$$\left.\int dkD^c(k)\int_0^{\nu} d\nu_1 e^{-iuk\nu_1}\int_0^{\nu} d\nu_2 e^{-iuk\nu_2}\right\}.$$

Noting now that the infinite constant c cancels in (25), we finally obtain the following expression for the Green's function

$$G(x-x') = i\int_0^{\infty} d\nu \exp\{-ip(x-x')$$

$$- i(m-up-i\varepsilon)\nu + f(\nu)\}, \quad (31)$$

where

$$f(\nu) \quad (32)$$

$$= \exp\left\{-\frac{ie^2}{2(2\pi)^3}\int dkD^c(k)\int_0^{\nu} d\nu_1 e^{-iuk\nu_1}\int_0^{\nu} d\nu_2 e^{iuk\nu_2}\right\}.$$

Since $D^c(k)$ is even with respect to k, the last formula, (32), can be put in the form

$$f(\nu) = \exp\left\{-\frac{ie^2}{(2\pi)^3}\int dkD^c(k)\int_0^{\nu} d\nu_1\int_0^{\nu_1} d\nu_2 e^{iuk\nu_2}\right\}, \quad (33)$$

In momentum space, the Green's function becomes

$$G(p) = i\int_0^\infty d\nu \exp\{i(up - m + i\varepsilon)\nu + f(\nu)\}. \tag{34}$$

Taking, then, $u = p/|p|$, the Green's function is

$$G(p) = i\int_0^\infty d\nu \exp\{i(|p| - m + i\varepsilon)\nu + f(\nu)\}. \tag{35}$$

4. RENORMALIZATION OF THE GREEN'S FUNCTION

There are infinities in the expression we have obtained for the Green's function. We use the Pauli-Villars regularization method [11], with one auxiliary mass M, to renormalize. To deal with the infrared catastrophe we introduce also the photon "mass" λ. Then we adopt the photon Green's function

$$D^c(k) = (\lambda^2 - k^2 - i\varepsilon)^{-1} - (M^2 - k^2 - i\varepsilon)^{-1}. \tag{36}$$

At the end of the calculation we must go to the limits $M \to \infty$ and $\lambda \to 0$. Hence in carrying out the integration (33), it is sufficient to obtain asymptotic values for $\lambda \sim 0$.

Carrying out the integration on ν_1 and ν_2, we obtain for $f(\nu)$ the expression

$$\begin{aligned} f(\nu) = &-\frac{ie^2}{(2\pi)^3}\int dk\left(\frac{1}{\lambda^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)\frac{1}{(uk)^2} \\ &+ \nu\frac{e^2}{(2\pi)^3}\int dk\left(\frac{1}{\lambda^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)\frac{1}{uk} \\ &+ \frac{ie^2}{(2\pi)^3}\int dk\left(\frac{1}{\lambda^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)\frac{e^{iuk\nu}}{(uk)^2}. \end{aligned} \tag{37}$$

The last integral remains finite as $M \to \infty$, as does the second for $\lambda \to 0$, so in these last two we can go to the indicated limits immediately. All the integrals can be calculated easily using the formula

$$x^{-1} = -i\int_0^\infty e^{ix\alpha - \varepsilon\alpha}d\alpha \tag{38}$$

and integrating first over k, then over the auxiliary variable α. The result is

$$f(\nu) = -(ie^2/2)M\nu + (e^2/\pi)\ln(M/\lambda) + \frac{e^2}{\pi}\ln(\nu\lambda). \tag{39}$$

We note that λ cancels out and the result is finite at $\lambda = 0$.

The first term, proportional to $i\nu$, gives the radiative correction to the mass of the electron, and diverges linearly as $M \to \infty$. It can be included in the effective mass of the electron, the renormalized mass being

$$m_1 = m + (e^2/2\pi)M. \tag{40}$$

We write that part, $\tilde{f}(\nu)$, of $f(\nu)$ which remains after mass renormalization in the form

$$\tilde{f}(\nu) = (e^2/\pi)\ln(M/m_1) + (e^2/\pi)\ln(m_1\nu). \tag{41}$$

The term independent of ν diverges logarithmically as $M \to \infty$, and determines the renormalization constant for the Green's function

$$Z = (M/m_1)^{e^2/\pi}. \tag{42}$$

Hence renormalization has removed all infinities from the theory and we obtain the following renormalized Green's function $G_1(p)$:

$$G_1(p) = Z^{-1}G(p), \tag{43}$$

$$\begin{aligned} G_1(p) &= i(m_1)^{e^2/\pi}\int_0^\infty d\nu \exp\{i(|p| - m_1 + i\varepsilon)\nu\}\nu^{e^2/\pi}. \end{aligned} \tag{44}$$

To evaluate the integral we make the substitution

$$x = |m_1 - |p||\nu, \tag{45}$$

Neglecting terms of order e^2/π, we finally obtain

$$G_1(p) = (m_1 - |p|)^{-1}|1 - |p|/m_1|^{-e^2/\pi}. \tag{46}*$$

In conclusion, I should like to express my deep gratitude to N. N. Bogoliubov, under whose guidance the present work was done.

*Selection of the coupling $D^c_{\alpha\beta}$ in transverse form would have led to a factor 3/2 in the exponent of (46). This can be shown with the aid of a gauge transformation.

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A Method of Obtaining Nonstationary Solutions of Boltzmann's Kinetic Equation

I. I. MOISEEV-OLKHOVSKII
Moscow State University

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A method is considered for obtaining nonstationary solutions of Boltzmann's kinetic equation. This method is free from the limitations of the Chapman-Enskog method. By way of an example, the dispersion of plane sound waves in a monatomic gas is considered.

IN recent times the most widely used method of solution of the Boltzmann equation has been the method of Chapman-Enskog[1]. However, this method is not applicable for a series of problems. Let us consider, for example, one such problem -- the problem of the dispersion of plane sound waves in a monatomic gas without consideration of the internal degrees of freedom of the atoms. Let the frequency of vibration of the external source be sufficiently low in comparison with the "characteristic frequency" of the gas, i.e., in comparison with the mean frequency of atomic collisions. Such a problem can be solved by making use of the equations of Navier-Stokes and Burnett[2], i.e., by making use of different approximations than those of the Chapman-Enskog method. If the frequency of vibration of the external source is comparable with or larger than the "characteristic frequency" of the gas, then these approximations lose their meaning. Actually, we use as the small parameter of the method of successive approximation employed in obtaining the equations of Navier-Stokes, Burnett, the ratio $\Delta t_p / \Delta t$, where Δt is a characteristic time interval for the process under consideration, for example, the period of vibration, Δt_p is the relaxation time of the gas. Thus we have as the condition of applicability of the Chapman-Enskog method the relation

$$\Delta t_p / \Delta t \ll 1. \qquad (1)$$

Another method is necessary, consequently, to obtain a solution differing widely from the quasi-equilibrium solution of the Boltzmann equation. Such a possibility is given by the method of "moments" (see, for example, Ref. 3).

Let us formulate a modification of this method. We use as the zeroth approximation the stationary solution of the Boltzmann equation. Then the first approximation gives the deviation of the density, velocity and temperature from the stationary distribution, and also the corresponding viscous force and heat flow. As conditions for the application of this method of small perturbations we have the relations

$$\frac{\Delta\rho}{\rho_0};\ \frac{\Delta u_i}{c_0};\ \frac{\Delta\Theta}{\rho_0 c_0^2};\ \frac{p_{ij}}{\rho_0 c_0^2};\ \frac{S_i}{\rho_0 c_0^3} \ll 1, \qquad (2)$$