Renormalizability of Pseudoscalar meson Theory with Pseudovector coupling

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It is shown that the convergence of a pseudoscalar theory with pseudovector coupling is not improved even when corrections for the propagation function for the mesons (which are connected with the vacuum polarization) are taken into account.

AS is well known, the program of renormalization, which has been successfully applied in quantum electrodynamics, is not sufficient to obtain convergent results that have physical meaning in most types of meson theories.[1] The hypothesis has been advanced (Ref. 2,* see also Ref. 3) that part of the divergences which take place in meson theories results only from the inapplicability of perturbation theory and that the convergence of pseudoscalar meson theory with pseudovector coupling can be improved if one were to use a true propagation function for the meson (i.e., to consider for the propagation function the energy corrections connected with the polarization of the nucleonic vacuum). Calculation of this function reduces to the calculation of the "polarization operator."

The authors of the works mentioned above make use of the method suggested by Feynman[4] for consideration of the vacuum polarization in quantum electrodynamics, which is practically identical with the regularization of Pauli-Villars. In momentum representation, the propagation function of the free meson is, as is known, equal to $(k^2+\mu^2)^{-1}$ (k is the external momentum). Improvement of the convergence of the theory is obtained as a consequence of the fact that the polarization operator computed by the authors mentioned varies as k^4 for $k\to\infty$. Making use of several general properties of the theory, it is possible to show that the polarization operator varies as k^2 as $k\to\infty$. Consequently, the result obtained in Refs. 2, 3 is based on a loose application of the Feynman method to the problem under consideration.

Thus, use of a true propagation function does not improve the convergence of the given theory. This is in agreement with the general results obtained in Ref. 5; weighty objections against the work of Ref. 2 were also put forward by Feldman.[6] Thus there is left in principle only the possibility of improvement of the convergence of the theory by use of a true "vertex part" (see Ref. 7).

1. We shall show that for the pseudoscalar mesonfield with pseudovector coupling, a certain relation is valid which we shall for brevity call the "theorem of equivalence", in a certain sense analogous to the law of continuity in quantum electrodynamics.**

Following Schwinger,[9] we define the Green's function of the spinor field:

$$G_{\alpha\beta}(x,x') = i\langle T(\psi_\alpha(x)\,\overline{\psi}_\beta(x'))\rangle.$$

Making use of the equations for ψ and $\bar\psi$, we obtain the following equation for $G(x,x')$:

$$\begin{gathered}\Big\{\gamma_\mu\frac{\partial}{\partial x_\mu} - ig\gamma_5\gamma_\mu\langle\frac{\partial\varphi}{\partial x_\mu}\rangle + m \\ - g\gamma_5\gamma_\mu\Big[\frac{\partial}{\partial z_\mu}\frac{\delta}{\delta J(z)}\Big]_{z\to x}\Big\}G(x,x') = \delta(x-x'),\end{gathered} \tag{1}$$

$$\begin{gathered}G(x,x')\Big\{-\gamma_\mu\frac{\partial}{\partial x'_\mu} + ig\langle\frac{\partial\varphi}{\partial x'_\mu}\rangle\gamma_\mu\gamma_5 + m \\ + g\Big[\frac{\partial}{\partial z_\mu}\frac{\delta}{\delta J(z)}\Big]_{z\to x'}\gamma_\mu\gamma_5\Big\} = \delta(x-x'),\end{gathered}$$

where J is the external source of the meson field. Making obvious transformations, we get

$$\mathrm{Sp}\Big\{\gamma_5\gamma_\mu\frac{\partial}{\partial x_\mu}G(x,x)\Big\} + 2m\,\mathrm{Sp}\{\gamma_5 G(x,x)\} = 0. \tag{2}$$

In the momentum representation this relation has the form

$$\int(dp)\,\mathrm{Sp}\{\gamma_5 i\hat{k}G(p,p-k)\} + 2m\int(dp)\,\mathrm{Sp}\{\gamma_5 G(p,p-k)\} = 0, \tag{3}$$

*It should be noted that charge renormalization is carried out incorrectly in Ref. 2.

**This relation is an immediate generalization of a similar relation obtained by Schwinger[8] without account of the proper radiation field.

where

$$(dp) = (2\pi)^{-2}\, d^4p; \quad G(p, p') = \int e^{-ipx} G(x, x')\, e^{ip'x'} (dx)(dx'). \tag{4}$$

The equation for the Green's function of the meson field has the form:

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} - \mu^2\right) D(x, x') + g\frac{\partial}{\partial x_\mu}\mathrm{Sp}\left\{\gamma_5\gamma_\mu \frac{\delta G(x, x)}{\delta J(x')}\Big|_{J=0}\right\} = -\delta(x - x'), \tag{5}$$

where

$$D(x, x') = \delta\varphi(x) / \delta J(x')\,|_{J=0}.$$

The polarization operator is defined by the following relation:

$$\int P(x, y)\, D(y, x')\, d^4y = -g\frac{\partial}{\partial x_\mu}\mathrm{Sp}\left\{\gamma_5\gamma_\mu\frac{\delta G(x, x)}{\delta J(x')}\Big|_{J=0}\right\}. \tag{6}$$

From this it follows that

$$P(x, x') = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x'_\nu} P_{\mu\nu}(x, x'), \tag{7}$$

where

$$P_{\mu\nu}(x, x') = -ig^2\int d^4y\, d^4y'\, \mathrm{Sp}\{\gamma_5\gamma_\mu G(x, y)\, \gamma_\nu\Gamma_5(y, x', y')\, G(y', x)\},$$
$$-i\gamma_\mu(\partial/\partial x_\mu)\,\Gamma_5(t, x, y) = \delta G^{-1}(t, y)/\delta g\varphi(x). \tag{7a}$$

Taking the equivalence relation (2) into account, we find that $P(x,x')$ is

$$P(x, x') = \frac{\partial}{\partial x'_\nu}\left\{2img^2\int d^4y\, d^4y' \times \mathrm{Sp}\,[\gamma_5 G(xy)\, \gamma_\nu\Gamma_5(y, x', y')\, G(y', x)]\right\}. \tag{8}$$

2. In the first approximation of perturbation theory,

$$\Gamma_5(tsy) = \gamma_5\delta(t - s)\,\delta(t - y).$$

In accord with Eqs. (7) and (7a),

$$P(k) = \frac{ig^2}{(2\pi)^4}\int d^4p\, \mathrm{Sp}\left\{\gamma_5\hat{k}\frac{1}{i(\hat{p} + \hat{k}) + m}\gamma_5\hat{k}\frac{1}{i\hat{p} + m}\right\}$$
$$= \frac{-8im^2g^2}{(2\pi)^4}\int d^4p\, \frac{k^2}{(p^2 + m^2)(p - k)^2 + m^2} + \frac{ig^2}{(2\pi)^4}\int d^4p\left\{\hat{k}\frac{1}{i\hat{p} + m}\hat{k}\frac{1}{i(\hat{p} - \hat{k}) + m}\right\}. \tag{9}$$

It follows from Eq. (8) that

$$P(k) = -\frac{8im^2g^2k^2}{(2\pi)^4}\int\frac{d^4p}{(p^2 + m^2)(p - k)^2 + m^2}. \tag{9a}$$

Comparing Eqs. (9) and (9a), we find, in the first approximation of perturbation theory, the following consequence of the theorem of "equivalence;"

$$\int d^4p\,\mathrm{Sp}\left\{\hat{k}\frac{i\hat{p} - m}{p^2 + m^2}\hat{k}\frac{i(\hat{p} - \hat{k}) - m}{(p - k)^2 + m^2}\right\} = 0. \tag{10}$$

This integral also figures in quantum electrodynamics, where it is equal to zero beçause of the law of continuity.

However, one can verify this fact by direct calculations only with the aid of artificial procedures (the regulators of Pauli-Villars, the "s-parametization" of Schwinger[8]). This is connected with the fact that all methods of calculating divergent integrals that are applied in quantum field theory (introduction of a shape factor, integration over an invariant finite region) are not invariant to a change of variable, or more precisely, to its "displacement", since $\int f(p-k)\, d^4p$ and $\int f(p)\, d^4p$ differ by some finite quantity in the case when the integrals are quadratic. Meanwhile, repeating the argument of the "theorem of equivalence" in momentum space, it is easy to convince oneself that the possibility of such a displacement of variable of integration is substantial.*

*The necessity of a displacement of the variable to reduce the integral of Eq. (10) to zero is also seen from the following:

$$\int\mathrm{Sp}\left\{\gamma_\mu\frac{1}{i\hat{p} + m}\hat{k}\frac{1}{i(\hat{p} - \hat{k}) + m}\right\}d^4p$$
$$= -i\int\mathrm{Sp}\left\{\gamma_\mu\frac{1}{i\hat{p} + m}[i\hat{p} + m - i(\hat{p} - \hat{k}) - m] \times \frac{1}{i(\hat{p} - \hat{k}) + m}\right\}d^4p$$
$$= -i\int\mathrm{Sp}\left\{\gamma_\mu\frac{1}{i(\hat{p} - \hat{k}) + m} - \gamma_\mu\frac{1}{i\hat{p} + m}\right\}d^4p = 0,$$

if we replace p by $p + k$ in the first term of the difference.

We shall start out from Eq. (9a) for the polarization operator. We subtract from the integral in this expression its value for $k = 0$:

$$\int \frac{d^4p}{(p^2+m^2)[(p-k)^2+m^2]} - \frac{d^4p}{(p^2+m^2)^2} \tag{11}$$
$$= \int d^4p \frac{(2pk-k^2)}{(p^2+m^2)^2[(p-k)^2+m^2]}$$
$$= 2\pi^2 i\{1-\theta \operatorname{ctg}\theta\},$$

where we have set $k^2 = -4m^2 \sin^2 \theta$. Consequently,

$$P(k) = 16\pi^2(2\pi)^{-4} m^2 g^2 k^2 \times (1-\theta\operatorname{ctg}\theta) + k^2 A, \tag{11a}$$

where $A = -8i(2\pi)^{-4} m^2 g^2 \int (p^2+m^2)^{-2} d^4p$ is a logarithmically diverging constant.

After renormalization, (9k) takes the form:

$$P'(k) = P(k) - P(k)\big|_{k^2=-\mu^2} - (k^2+\mu^2)\frac{\partial P}{\partial k^2}\bigg|_{k^2=-\mu^2} \tag{12}$$
$$= -\frac{g^2 m^6 (k^2+\mu^2)^2}{(2\pi)^2}\int_0^1 \left\{v^2 dv \Big/ \left[m^2 + \frac{k^2}{4}(1-v^2)\right]\left[m^2 - \frac{\mu^2}{4}(1-v^2)\right]^2\right\}$$
$$= \frac{4m^2 g^2}{(2\pi)^2}\left\{k^2(\theta_0 \operatorname{ctg}\theta_0 - \theta\operatorname{ctg}\theta) + (k^2+\mu^2)\left(\frac{1}{2} - \frac{\theta_0}{\sin 2\theta_0}\right)\right\},$$

where $\mu^2 = 4m^2 \sin^2 \theta_0$.

3. Upon calculation of the polarization operator by Feynman's method, we get

$$P(k) = -8i(2\pi)^{-4} g^2 k^2 2\pi^2 i \times \{1-\theta\operatorname{ctg}\theta + k^2/12m^2\} + k^2 C,$$

where C is a logarithmically diverging constant. The difference from our result (11a) is associated with the fact that in the Feynman method*

$$\int m^2 f(p)\, d^4p \neq m^2 \int f(p)\, d^4p.$$

In electrodynamics, this does not lead to non-single-valuedness in the calculation of the finite part of the polarization operator, since it manifests itself only in the change of the renormalization constant. In our case also, because of the fact that $P(k)$ has a positive factor k^2 in comparison with electrodynamics, this non-single-valuedness is not removed by the renormalization.

The usual program of renormalization consists of the separation from the divergent expression of the part which contains some first powers of the external momentum, while the remainder no longer contains divergences, and the diverging parts are removed as a consequence of the renormalization of mass and charge. For an expression which diverges logarithmically, this program can be carried out effectively. But in the case of the proper mass of the photon or meson, the initial expression diverges quadratically. A stronger divergence leads to an impossibility of separating the divergent part without introduction or artificial procedure, and the available natural methods violate the general properties of the theory (such as the law of continuity in quantum electrodynamics, or the "theorem of equivalence" in the case of pseudoscalar meson theory with pseudovector coupling).

We have shown that, using the "theorem of equivalence", it is possible to represent the polarization operator in the form of an expression which diverges logarithmically, for which the program of renormalization can be effectively carried out without use of any artificial prodedures.

The reason for the difficulty in the calculation of the vacuum polarization both in quantum electrodynamics and in our case lies in the fact that, in the determination of the polarization operators (see Ref. 5), there enters $G(x,x)$, understood as the limit

$$G(x,x) = \lim_{\xi\to 0} {}^1\!/_2\{G(x, x+\xi) + G(x+\xi, x)\}.$$

However, this limit is insufficiently defined, since the direction has not been indicated along which ξ approaches zero. As a unique method of definition of this process ($\xi \to 0$) there is the requirement that the limit process not violate the general properties which follow from the basic equation of the theory. The condition of

*Our result corresponds to the fact that the regularization of Feynman is applied to

$$\int \frac{d^4p}{(p^2+m^2)[(p-k)^2+m^2]},$$

and not to $\displaystyle\int \frac{m^2 d^4p}{(p^2+m^2)[(p-k)^2+m^2]}$.

gauge-invariance in quantum electrodynamics and the "theorem of equivalence" in our case will, as we shall show, be satisfied if the limit process satisfies the condition $k\ \xi = 0$, where k is the external momentum of the photon or meson.

In fact, let us consider the polarization operator which corresponds to an "open loop" of the meson eigen energy:

$$P\left(x+\frac{\xi}{2};\ x-\frac{\xi}{2};\ x'\right)$$

$$= g^2 \frac{\partial}{\partial x_\mu} \mathrm{Sp}\left\{\gamma_5\gamma_\mu \frac{\delta G\,(x+\xi/2,\, x-\xi/2)}{\delta g\varphi\,(x')}\right\}.$$

In the momentum representation,

$$P(k,\xi) = \frac{ig^2}{(2\pi)^4}\int e^{ip\xi}\, d^4p$$

$$\times\ \mathrm{Sp}\left\{\gamma_5\hat{k}\,\frac{1}{i(\hat{p}+\hat{k}/2)+m}\,\gamma_5\hat{k}\,\frac{1}{i(\hat{p}-\hat{k}/2)+m}\right\}.$$

Here the polarization operator of interest to us, $P\ (k)$, is the limit of $P\ (\ k,\ \xi\ )$ as $\xi \to 0$. Just as in Eq. (9) we divide $P\ (k,\ \xi)$ into two parts:

$$P(k,\xi) = P_1(k,\xi) + P_2(k,\xi);$$

$$P_1(k,\xi) = \frac{ig^2}{(2\pi)^4}\int d^4p e^{ip\xi}\,\mathrm{Sp}\left\{\hat{k}\,\frac{1}{i(\hat{p}+\hat{k}/2)+m}\,\hat{k}\,\frac{1}{i(\hat{p}-\hat{k}/2)+m}\right\};$$

$$P_2(k,\xi) = -\frac{8im^2g^2k^2}{(2\pi)^4}\int d^4p e^{ip\xi}\,\frac{1}{[(p+k/2)^2+m^2]\,[(p-k/2)^2+m^2]}\,.$$

The integral which enters into $P_1(k,\ \xi)$ is computed in Ref. 10 and leads to

$$P_1(k,\xi) = -4ig^2$$

$$\times\left\{k_\mu \frac{\partial}{\partial \xi_\mu}\,\Delta + (\xi)\int_{-1}^{1} dv\,(k,\xi)\, e^{iv\,(k,\xi)/2}\right\}.$$

If we superimpose on the limiting process $\xi \to 0$ the condition $k\,\xi\ = 0$, then $P_1\ (k,\ \xi\ )$ reduces to zero in correspondence with the theorem of equivalence.*

In analogy with (11), we remove from the integral entering into $P_2\ (k,\ \xi)$ its value for $k = 0$:

$$\int d^4p\,\frac{e^{ip\xi}}{[(p+k/2)^2+m^2]\,[(p-k/2)^2+m^2]} - \int d^4p\,\frac{e^{ip\xi}}{(p^2+m^2)^2}$$

$$= \int d^4p\,\frac{e^{ip\xi}\,(k^2/2)\ (p^2-m^2-k^2/8)}{[(p+k/2)^2+m^2]\,[(p-k/2)^2+m^2]\,(p^2+m^2)}\,.$$

This integral has no singularities for $\xi = 0$ and is equal to $2\ \pi^2\ i\ (1-\theta\ \mathrm{ctg}\ \theta\ )$ in this case. After renormalization, we again obtain the value (12) for $P\ (\ k\ )$.

[1] P. Matthews and A. Salam, Revs. Mod. Phys. **23**, 311 (1951).

[2] Ning Hu, Phys. Rev. **80**, 1109 (1950).

[3] A. D. Galanin, J. Exptl. Theoret. Phys. (U.S.S.R.) **26**, 417 (1954).

[4] R. Feynman, Phys. Rev. **76**, 769 (1949).

[5] R. Lehmann, Nuovo Cimento **11**, 342 (1954).

[6] D. Feldman, Proc. Roy. Soc. (London) **A223**, 113 (1954).

[7] S. F. Edwards, Phys. Rev. **90**, 284 (1953).

[8] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[9] J. Schwinger, Proc. Nat. Acad. Sci. **37**, 452 (1951).

[10] J. Valatin, Proc. Roy. Soc. (London) **222A**, 228 (1954).

*This is also seen from the following:

$$\int d^4pe^{ip\xi}\,\mathrm{Sp}\left\{\hat{k}\,\frac{1}{i(\hat{p}+\hat{k}/2)+m}\,\hat{k}\,\frac{1}{i(\hat{p}-\hat{k}/2)+m}\right\}$$

$$= \int d^4pe^{ip\xi}\,\mathrm{Sp}\left\{\hat{k}\,\frac{1}{i(\hat{p}+\hat{k}/2)+m} - \hat{k}\,\frac{1}{i(\hat{p}+\hat{k}/2)+m}\right\}$$

$$= \int d^4pe^{i\,(p\xi+k\xi/2)}\,\mathrm{Sp}\left\{\hat{k}\,\frac{1}{i\hat{p}+m}\right\}$$

$$- \int d^4pe^{i\,(p\xi+k\xi/2)}\,\mathrm{Sp}\left\{\hat{k}\,\frac{1}{i\hat{p}+m}\right\} = 0,$$

if $k\ \xi = 0$. The validity of the change of variable here is not called into question since all the integrals converge.

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