A Contribution to the General Theory of Statistical Equilibrium of a System of Interacting Particles

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Equations for the equilibrium distribution functions are obtained from the general dynamic equations, without making the assumption of the canonical character of the initial equilibrium distribution. In place of this a series of physical conditions are imposed of the type of the condition of diminishing correlations with increase of the distances between the particles.

IF, for systems consisting of N similar interacting particles occupying a volume V, we postulate the existence of a class of distribution functions[1,2]

$$n_s(t, \mathbf{x}^1, \ldots, \mathbf{x}^s), \quad (1)$$

$$f_s(t, \mathbf{x}^1 \ldots, \mathbf{x}^s, \mathbf{y}^1, \ldots, \mathbf{y}^s),$$

$$g_s(t, \mathbf{x}^1, \ldots, \mathbf{x}^s, \mathbf{y}^1, \ldots, \mathbf{y}^s, \mathbf{z}^1, \ldots, \mathbf{z}^s), \ldots$$

($\mathbf{x}^i$ is a vector determining the position of the particles, $\mathbf{y}^i$ a velocity vector of the particles, $\mathbf{z}^i$ an acceleration vector, etc.), and if we subject these distribution functions to the conditions

$$\int \ldots \int f_s(t, \mathbf{x}, \mathbf{y}) \prod_{i=1}^{s} d\mathbf{y}^i = n_s(t, \mathbf{x}), \quad (2)$$

$$\int \ldots \int g_s(t, \mathbf{x}, \mathbf{y}, \mathbf{z},) \prod_{i=1}^{s} d\mathbf{z}^i = f_s(t, \mathbf{x}, \mathbf{y}),$$

$$\int \ldots \int f_{s+1}\, d\mathbf{x}^{s+1}\, d\mathbf{y}^{s+1} = A f_s, \quad (3)$$

($\mathbf{x}$ and $\mathbf{y}$ denote, respectively, the sets of vectors $\mathbf{x}^1, \ldots, \mathbf{x}^s$ and $\mathbf{y}^1, \ldots, \mathbf{y}^s$; A is a normalizing factor), and if, besides, we require fulfillment of the equations

$$\lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \quad (4)$$

$$\times \left\{ \int \ldots \int g_s(t - \delta\tau, \mathbf{x} - \mathbf{y}\delta\tau, \mathbf{y} - \mathbf{z}\delta\tau, \mathbf{z}) \prod_{i=1}^{s} d\mathbf{z}^i - \int \ldots \int g_s(t, \mathbf{x}, \mathbf{y}, \mathbf{z}) \prod_{i=1}^{s} d\mathbf{z}^i \right\} = 0$$

and of anologous equations for the other distribution functions*, then it is possible to obtain a system of basic equations of the kinetic theory

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \frac{\partial f_s}{\partial \mathbf{x}^i} \mathbf{y}^i = \frac{1}{m} \sum_{i,j}^{s} \frac{\partial f_s}{\partial \mathbf{y}^i} \frac{\partial \Phi^{ij}}{\partial \mathbf{x}^i} \quad (5)$$

$$+ \frac{1}{m} \sum_{i=1}^{s} \int\int \frac{\partial \Phi^{i,\, s+1}}{\partial \mathbf{x}^i} \frac{\partial f_{s+1}}{\partial \mathbf{y}^i} d\mathbf{x}^{s+1}\, d\mathbf{y}^{s+1}$$

in the absence of external forces, for an interaction law given by a mutual potential Φ.

As we demonstrated[2], N. N. Bogoliubov's equations for the system under consideration, the Maxwell distribution, and (for $s=N$) the Gibbs distribution all follow from Eq. (5), in the determination of the equilibrium state as well as of the stationary state of a system in which the distribution function f_s factors into the product of two independent functions giving, respectively, the distributions of the velocities and of the coordinates [$f_s = \psi_s(\mathbf{x})\, \varphi_s(\mathbf{y})$]. A generalization of the results[2] in the case of complex systems, consisting of particles of different kinds, has also been derived[3].

In constructing a kinetic theory independent, in a certain sense, of the results of statistical mechanics, the theorem proved in Refs. 2 and 3 should be considered as a sufficient condition for obtaining an equilibrium solution. It is necessary to prove the necessity of the above-indicated properties of the equilibrium distribution functions for the dynamic states of complex particles. It is desirable to derive this proof with the least possible use of special assumptions about the character of the law of interaction between the particles.

Let us write Eq. (5) in the form[3,4]

$$\frac{\partial F_s}{\partial t} = [H_s;\, F_s] \quad (6)$$

$$+ \frac{1}{v} \int \left[\sum_{i=1}^{s} \Phi_{i,\, s+1};\; F_{s+1} \right] d\mathbf{q}_{s+1}\, d\mathbf{p}_{s+1},$$

* In the work of Born and Green[1] and in our previous article[2] this condition is inaccurately written. The correct expression is the one presented above.

where F_s represents the function f_s, normalized in the manner of Bogoliubov, and where the momenta appear in place of the velocities.

Let us assume that the density of our system is small enough that the expansion

$$F_s = F_s^0 + v^{-1}F_s^1 + v^{-2}F_s^2 + \ldots \qquad (7)$$

is meaningful. Considering henceforth a system in the absence of external forces, for the sake of simplicity, we shall look for solutions of the equations

$$[H_s;\ F_s] + \frac{1}{v}\int\left[\sum_{i=1}^{s}\Phi_{i,\,s+1},\ F_{s+1}\right]d\mathbf{q}_{s+1}\,d\mathbf{p}_{s+1} = 0, \qquad (8)$$

for the stationary state, subject to the conditions (7), assuming central forces of interaction. Besides the conditions enumerated in Eqs. (2)-(4), the equilibrium distribution functions must satisfy boundaryconditions for relaxation of the correlations, having the form

$$F_s(\mathbf{q}_1, \ldots, \mathbf{p}_s) \to \prod_{1\leqslant i\leqslant s} F_1(\mathbf{q}_i, \mathbf{p}_i) \qquad (9)$$

for all $|\mathbf{q}_i - \mathbf{q}_j| \to \infty$.

Let us substitute the expansion (7) into Eq. (8) and consider the corresponding systems of equations. For the zero-order approximation of the unary function (single-particle distribution) we obtain

$$m^{-1}(\mathbf{p}_1 \operatorname{grad} F_1^0) = 0, \qquad (10)$$

from which it follows that F_1^0 can depend only on $\mathbf{p}_1$.

For the zero-order approximation to the s-dimensional distribution function, we have the following equation

$$\sum_{\alpha=1}^{3}\left\{\frac{\partial U_s}{\partial q_1^\alpha}\frac{\partial F_s^0}{\partial p_1^\alpha} + \ldots + \frac{\partial U_s}{\partial q_s^\alpha}\frac{\partial F_s^0}{\partial p_s^\alpha} - \frac{1}{m}\left(p_1^\alpha\frac{\partial F_s^0}{\partial q_1^\alpha} + \ldots + p_s^\alpha\frac{\partial F_s^0}{\partial q_s^\alpha}\right)\right\} = 0, \qquad (11)$$

and conditions on the relaxation of the correlations of the form

$$F_s^0 - \prod_{1\leqslant i\leqslant s} F_1^0 \to 0$$

for all $|\mathbf{q}_i - \mathbf{q}_j| \to \infty$. The equations of the characteristics

$$\frac{dq_i^\alpha}{\partial H_s/\partial p_i^\alpha} = \frac{dp_i^\alpha}{-\partial H_s/\partial q_i^\alpha},$$

$$\alpha = 1, 2, 3;\ i = 1, \ldots, s$$

possess $6s - 1$ integrals of the motion: $J_k(\mathbf{q}_1, \ldots, \mathbf{p}_s) = c_k$.

Let us assume that the transition to the limit $|\mathbf{q}_i - \mathbf{q}_j| \to \infty$ is "adiabatic"; i.e., let us assume that in the passage to the limit to be considered the limiting functions F_s^0 depends on the same integrals of the motion as the initial function F_s^0. Since F_s^0 is an implicit function of $\mathbf{q}$ and $\mathbf{p}$,

$$\lim F_s^0(J_1, \ldots, J_{6s-1}) = F_s^0(\lim J_1, \ldots, \lim J_{6s-1}).$$

If we set $\lim J_k = J_k^*$ then, from the condition on the relaxation of the correlations, we obtain

$$F_s^0(J_1^*, \ldots) = F_1^0(\mathbf{p}_1)\ldots F_1^0(\mathbf{p}_s) \qquad (12)$$

or

$$\ln F_s^0(J_1^*, \ldots) = \sum_{i=1}^{s}\ln F_1^0(\mathbf{p}_i). \qquad (13)$$

Because of the fact that the F_s^0 are functions of the "s-body" problem, while the F_1^0 are functions of the "single-body" problem, it follows from (13) that in the limit $\ln F_s^0$ depends only on additive integrals of the motion and is made up of a linear combination of them. Not considering the uniform translational and rotational motion of the system as a whole, we obtain

$$\ln F_s^0(\mathbf{J}_1^*, \ldots) = \alpha_s - \beta W_s^*, \qquad (14)$$

where $W_s^* = \lim H_s$. In this way the solution sought for will have the form

$$F_s^0 = c_s \exp(-\beta H_s) \qquad (15)$$

in connection with the "adiabatic" character of the limiting transition being considered; in particular

$$F_1^0 = c_1 \exp(-\beta p_1^2/2m).$$

In agreement with the conditions on the relaxation of the correlations, $c_s = (c_1)^s$, and from the normalization conditions it follows that

$$c_1 = \left(\int \exp\left(-\beta p^2/2m\right) dp\right)^{-1}.$$

For the first approximation to the single-variable function

$$-m^{-1}(\mathbf{p}_1 \operatorname{grad} F_1^1) + \int [\Phi_{12};\ F_2^0] d\mathbf{q}_2\, d\mathbf{p}_2 = 0 \quad (16)$$

and since

$$\int \left[\Phi_{12};\ F_2^0\right] d\mathbf{q}_2 d\mathbf{p}_2$$
$$= \sum_{\alpha=1}^{3} c_2 \left[\int \exp\left(-\frac{\beta p_2^2}{2m}\right) d\mathbf{p}_2 \frac{p_1^\alpha}{m} \exp\left(-\frac{\beta p_1^2}{2m}\right) \times \frac{\partial}{\partial q_1^\alpha} \int \exp\left(-\beta\Phi_{12}\right) d\mathbf{q}_2 + \exp\left(-\frac{\beta p_1^2}{2m}\right) \int \frac{\partial}{\partial q_2^\alpha} \exp\left(-\beta\Phi_{12}\right) d\mathbf{q}_2 \times \int \frac{p_2^\alpha}{m} \exp\left(-\frac{\beta p_2^2}{2m}\right) d\mathbf{p}_2\right] = 0,$$

we deduce that F_1^1 can depend only on $\mathbf{p}_1$. From the normalization condition it further follows that $F_1^1 = 0$. Analogously, we find that all the higher approximations for the single-variable functions vanish.

Passing to the binary functions (joint distributions), let us examine the equation

$$[H_2;\ F_2^1] + \int \left[\sum_{i=1}^{2} \Phi_{i3};\ F_3^0\right] d\mathbf{q}_3 d\mathbf{p}_3 = 0; \quad (17)$$

Using (15) we can write

$$\int \left[\sum_{i=1}^{2} \Phi_{i3};\ F_3^0\right] d\mathbf{q}_3\, d\mathbf{p}_3 \quad (18)$$
$$= \sum_{\alpha} c_3 \exp\left(-\beta\Phi_{12}\right) \exp\left[-\frac{\beta}{2m}\left(p_1^2 + p_2^2\right)\right] \times \int \exp\left(-\frac{\beta p_3^2}{2m}\right) d\mathbf{p}_3 \left[\frac{p_1^\alpha}{m} \times \int \frac{\partial}{\partial q_1^\alpha} \exp\left[-\beta\left(\Phi_{13} + \Phi_{23}\right)\right] d\mathbf{q}_3 + \frac{p_2^\alpha}{m} \int \frac{\partial}{\partial q_2^\alpha} \exp\left[-\beta\left(\Phi_{13} + \Phi_{23}\right)\right] d\mathbf{q}_3\right].$$

On the other hand, setting

$$F_2^1 = F_2^0 \varphi_2^*, \quad (19)$$

for the particular solution of the inhomogeneous Eq. (17), we get

$$[H_2;\ F_2^1] = [H_2;\ F_2^0 \varphi_2^*] = [H_2;\ F_2^0]\,\varphi_2^* + [H_2;\ \varphi_2^*]\, F_2^0 = [H_2;\ \varphi_2^*]\, F_2^0; \quad (20)$$
$$[H_2;\ \varphi_2^*] + \sum_{\alpha} \left\{\frac{\partial\Phi_{12}}{\partial q_1^\alpha} \frac{\partial\varphi_2^*}{\partial p_1^\alpha} + \frac{\partial\Phi_{12}}{\partial q_2^\alpha} \frac{\partial\varphi_2^*}{\partial p_2^\alpha} - \frac{p_1^\alpha}{m} \frac{\partial\varphi_2^*}{\partial q_1^\alpha} - \frac{p_2^\alpha}{m} \frac{\partial\varphi_2^*}{\partial q_2^\alpha}\right\}.$$

Noting that

$$c_3 \int \exp\left(-\beta p^2/2m\right) d\mathbf{p} = c_2,$$

we find a particular solution in the form

$$F_2^0 \varphi_2^* = F_2^0 \int \exp\left[-\beta\left(\Phi_{13} + \Phi_{23}\right)\right] d\mathbf{q}_3. \quad (21)$$

The general solution of Eq. (17) is represented by the expression

$$F_2^1 = F_2^0 \varphi_2^* + b_2' F_2^0, \quad (22)$$

where the constant b_2^1 must be chosen so that F_2^1 satisfies the condition on the relaxation of the correlations: $F_2^1 \to 0$ for $|\mathbf{q}_i - \mathbf{q}_j| \to \infty$. Correspondingly b_2^1 is given by

$$b_2^1 = -\int \left(1 + f_{13} + f_{23}\right) d\mathbf{q}_3,$$

where $f_{ij} = \exp\left(-\beta\Phi_{ij}\right) - 1$, and F_2^1 can be represented in the form

$$F_2^1 = F_2^0 \int f_{13} f_{23} d\mathbf{q}_3. \quad (23)$$

For F_3^1 we obtain in an analogous manner

$$F_3^1 = F_3^0 \int \exp\left[-\beta\left(\Phi_{14} + \Phi_{24} + \Phi_{34}\right)\right] d\mathbf{q}_4 + b_3' F_3^1, \quad (24)$$

where

$$b_3^1 = -\int \left(1 + f_{14} + f_{24} + f_{34}\right) d\mathbf{q}_4.$$

Let us further consider the equations

$$[H_2;\ F_2^2] + \int \sum_{1 \leqslant i \leqslant 2} [\Phi_{i3};\ F_3^1]\, d\mathbf{q}_3\, d\mathbf{p}_3 = 0. \quad (25)$$

Substituting F_3^1 from (24) we obtain

$$[H_2; F_2^2] + \sum_\alpha F_2^0\Big[\frac{p_1^\alpha}{m}\int\frac{\partial}{\partial q_1^\alpha} \times \exp[-\beta(\Phi_{13}+\Phi_{23})]\,\varphi_3^* d\mathbf{q}_3 + \frac{p_2^\alpha}{m}\int\frac{\partial}{\partial q_2^\alpha}\exp[-\beta(\Phi_{13}+\Phi_{23})]\,\varphi_3^* d\mathbf{q}_3\Big] + b_3^1\sum_\alpha F_2^0\Big[\frac{p_1^\alpha}{m}\int\frac{\partial}{\partial q_1^\alpha}\exp[-\beta(\Phi_{13}+\Phi_{23})]\,d\mathbf{q}_3 + \frac{p_2^\alpha}{m}\int\frac{\partial}{dq_2^\alpha}\exp[-\beta(\Phi_{13}+\Phi_{23})]\,d\mathbf{q}_3\Big] = 0.$$

Seeking a particular solution of the inhomogeneous equation in the form $F_2^2 = F_2^0\varphi_2^{**}$, where φ_2^{**} does not depend on the momenta, we obtain

$$\frac{\partial\varphi_2^{**}}{\partial q_1^\alpha} = \int\frac{\partial}{\partial q_1^\alpha}\exp[-\beta(\Phi_{13}+\Phi_{23})] \times\Big\{\int\exp[-\beta(\Phi_{14}+\Phi_{24}+\Phi_{34})]\,dq_4\Big\}dq_3 + b_3^1\frac{\partial}{\partial q_1^\alpha}\int\exp[-\beta(\Phi_{13}+\Phi_{23})]\,d\mathbf{q}_3,$$

from which we find for the desired solution

$$F_2^0\varphi_2^{**} = \frac{1}{2}F_2^0\int\exp[-\beta(\Phi_{13}+\Phi_{23} + \Phi_{14}+\Phi_{24}+\Phi_{34})]\,d\mathbf{q}_3 d\mathbf{q}_4 + b_3^1F_2^0\int\exp[-\beta(\Phi_{13}+\Phi_{23})]\,d\mathbf{q}_3. \tag{26}$$

We can then write the general solution in the form

$$F_2^2 = F_2^0\varphi_2^{**} + b_2^2F_2^0 \tag{27}$$

and from our boundary conditions we obtain for the constant b_2^2 the expression

$$b_2^2 = \frac{1}{2}\int(1+f_{14}+f_{24}+f_{13}+f_{23} + f_{34}+f_{13}f_{14}+f_{13}f_{24}+f_{14}f_{23} + f_{23}f_{24}-f_{13}f_{34}f_{14}-f_{23}f_{34}f_{24})\,d\mathbf{q}_3 d\mathbf{q}_4. \tag{28}$$

Finally, F_2^2 is represented by the product

$$F_2^2 = F_2^0L_{12}^{34}, \tag{29}$$

where the symbol L_{12}^{34} designates a collection of integral terms determining all possible correlational relations between a fixed group of particles (1, 2) and the "running" group of particles (3, 4). In this fashion we have for the binary functions

$$F_2^0,\ F_2^1 = F_2^0L_{12}^3,\quad F_2^2 = F_2^0L_{12}^{34},\ \ldots$$

and for the ternary functions

$$F_3^0,\ F_3^1 = F_3^0L_{123}^4,\quad F_3^2 = F_3^0L_{123}^{45},\ \ldots$$

In connection with the fact that we are at the moment interested only in the question of the general structure of the solution (in the sense of its multiplicity with respect to the dependence on coordinates and momenta) and not in the problem of the exact construction of all the higher approximations, we shall examine only the "principal" parts of the particular solutions of the corresponding inhomogeneous equations, i.e,, expressions of the form

$$\overline{F}_s^i = \frac{F_s^0}{i!}\int\mu_s\mu_{s+1}\ldots\mu_{s+i-1}d\mathbf{q}_{s+1}\ldots d\mathbf{q}_{s+i}, \tag{30}$$

$$\mu_s = \exp\Big(-\beta\sum_{j=1}^{s}\Phi_{j,\,s+1}\Big);$$

where the superscript i indicates the order of the approximation. Postulating for $\overline{F}_{s+1}^{i-1}$ the form (30), let us consider the equation

$$[H_s; \overline{F}_s^i] + \int\Big[\sum_{j=1}^{s}\Phi_{j,\,s+1};\ \overline{F}_{s+1}^{i-1}\Big]d\mathbf{q}_{s+1}d\mathbf{p}_{s+1}. \tag{31}$$

Substituting $\overline{F}_s^i = (F_s^0/i!)\gamma_s^i$, we obtain

$$\frac{1}{i}\sum_\alpha F_s^0\Big\{-\frac{p_1^\alpha}{m}\frac{\partial\gamma_s^i}{\partial q_1^\alpha}-\ldots-\frac{p_s^\alpha}{m}\frac{\partial\gamma_s^i}{\partial q_s^\alpha}\Big\} + \sum_\alpha\int\Big\{\frac{\partial\Phi_{1,\,s+1}}{\partial q_1^\alpha}\frac{\partial F_{s+1}^0}{\partial p_1^\alpha} + \ldots+\frac{\partial\Phi_{s,\,s+1}}{\partial q_s^\alpha}\frac{\partial F_{s+1}^0}{\partial p_s^\alpha}\Big\}\gamma_{s+1}^{i-1}d\mathbf{q}_{s+1}d\mathbf{p}_{s+1}$$

Comparing corresponding terms, we can write:

$$\int\frac{\partial}{\partial q_1^\alpha}(\mu_s)\int\mu_{s+1}\ldots\mu_{s+i-1}\,d\mathbf{q}_{s+1}\,d\mathbf{q}_{s+2}\ldots d\mathbf{q}_{s+i} = \frac{1}{i}\frac{\partial\gamma_s^i}{\partial q_1^\alpha}$$

and since

$$\frac{\partial\mu_s}{\partial q_1^\alpha}\mu_{s+1}\ldots\mu_{s+i-1}d\mathbf{q}_{s+1}\ldots d\mathbf{q}_{s+i} = \frac{1}{i}\frac{\partial}{\partial q_1^\alpha}\int\mu_s\mu_{s+1}\ldots\mu_{s+i-1}\,d\mathbf{q}_{s+1}\ldots d\mathbf{q}_{s+i},$$

we arrive at the result

$$\gamma_s^i = \int \mu_s \mu_{s+1} \ldots \mu_{s+i-1} d\mathbf{q}_{s+1} \ldots d\mathbf{q}_{s+i}. \qquad (32)$$

The above reasoning serves as a proof by induction of the correctness of the expression (30) for the "principal" parts of the particular solutions $\overline{F}_s^i$. $\overline{F}_s^i$ differs from the exact solution F_s^i only that in the latter additional terms are contained, while the structure of the solution $\overline{F}_s^i$ coincides with the structure of F_s^i, in the sense in which we are interested. To determine the exact solution F_s^i it is always necessary to operate not with the "principal" parts of the particular solutions, but with the complete solutions, i.e., with expressions of the type of Eq. (27), in which the constants b_s^k are chosen with reference to the boundary conditions on the relaxation of the correlations, as was done in obtaining Eq. (28). Use of the general solutions does not essentially complicate the proof of the stated theorem and leads only to more cumbersome expressions. We note, by the way, that the procedure considered by us also gives another method for the construction of the results of the theory of Ursell and Maier for the coordinate part of the distribution function. All the reasoning presented above holds also in the presence of external conservative forces; in this case one must understand H_s the energy of the corresponding group of particles including these external forces.

We shall make a few remarks regarding the dependence of the applicability of the above considerations on the character of the interaction law $\Phi(r)$ between the particles and on the density of the system. If the interaction potential falls off with increasing r faster than r^{-3}, and if the density of the system is sufficiently small, the multiplicative structure of the equilibrium solution is that derived above, where in the second part of the theorem an essential role is played by the assumption of the adiabatic character of the transition to the limit in the condition of the relaxation of the correlations, in the sense indicated earlier.

But if the potential Φ falls off with increasing r as r^{-3} or more slowly, then the series representing the part of the solution depending on the coordinates diverges, and an additional investigation is necessary. For a potential of the form

$$\Phi(r) = -ar^{-n} + br^{-m},$$

$$(m > n) \quad \text{or} \quad \Phi(r) = -ar^{-n} + Ae^{-\alpha r}$$

the reasoning presented is correct for $n > 3$, i.e., for a "sphere of influence" of the intermolecular forces which is finite, but not necessarily infinitely small (small on the microscopic scale).

Consequently, while in the considerations of Green[1] the smallness of the density of the system is not used, but rather the microscopic smallness of the "sphere of influence" is used (in essence a mechanism of instantaneous collisions is assumed), in the theorem presented above the opposite situation obtains: the smallness of the density of the system is explicitly used, while the microscopic smallness of the sphere of influence of intermolecular forces is not required; the size of the sphere of influence is limited simply by the formal requirement of convergence of the integrals appearing as coefficients of a series in powers of the density. In order to relax the requirements referring to the density of the system and to the character of the interaction, as mentioned above, an additional investigation is necessary.

In conclusion, let us consider the example of a system on which, besides the conservative internal forces, external forces depending only on the velocities of the particles are also acting.

The basic equation for the distribution function f_s in the most general case of arbitrary forces has the form:

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \frac{\partial f_s}{\partial \mathbf{x}_i} \mathbf{y}_i + \frac{1}{m} \times \sum_{i=1}^{s} \frac{\partial}{\partial \mathbf{y}_i} (f_s \mathbf{F}_i) + \frac{1}{m} \sum_{i,j} \frac{\partial}{\partial \mathbf{y}_i} (f_s \mathbf{F}_{ij}) + \frac{1}{m} \int\int \sum_{i=1}^{s} \frac{\partial}{\partial \mathbf{y}_i} (\mathbf{F}_{i,s+1} f_{s+1})\, d\mathbf{x}_{s+1} d\mathbf{y}_{s+1} = 0, \qquad (33)$$

where we have used the same notation as in Ref. 3 and in Eqs. (1)-(5) of this article. Setting

$$\partial f_s / \partial t = 0, \quad f_s = \psi_s(\mathbf{x})\, \varphi_s(\mathbf{y}),$$

for the equilibrium solution, we obtain

$$\sum_{i=1}^{s} \left\{ \frac{\partial \psi_s}{\partial \mathbf{x}_i} \varphi_s \mathbf{y}_i + \frac{1}{m} \frac{\partial \varphi_s}{\partial \mathbf{y}_i} \left(\sum_{j=1}^{s} \psi_s \mathbf{F}_{ij} + \int \psi_{s+1} \mathbf{F}_{i,s+1} d\mathbf{x}_{s+1} \right) + \frac{\psi_s}{m} \left(\frac{\partial \varphi_s}{\partial \mathbf{y}_i} \mathbf{F}_i + \frac{\partial \mathbf{F}_i}{\partial \mathbf{y}_i} \varphi_s \right) \right\} = 0. \qquad (34)$$

A necessary condition for the realization of an

equilibrium solution, existing in the absence of external forces of the type considered above, is the simultaneous fulfillment of the following equations

$$\frac{\partial \varphi_s}{\partial y_i}\frac{\alpha}{m} = \varphi_s y_i; \quad \frac{\partial \varphi_s}{\partial y_i} F_i + \frac{\partial F_i}{\partial y_i}\varphi_s = 0 \qquad (35)$$

and, consequently, the forces F_i must satisfy the condition

$$(m/\alpha)\,(y_i F_i) + \partial F_i / \partial y_i = 0. \qquad (36)$$

In particular it follows from (36) that gyroscopic forces will not disturb the equilibrium condition of the system. In an analogous way one can treat the more general conditions mentioned in Ref. 2.

I take this occasion to express my gratitude to N. N. Bogoliubov for proposing the problem, and also to M. A. Leontovich and Ia. B. Lopatinskii for discussion of questions connected with the work.

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Electron Energy Spectrum in a Crystal Located in a Magnetic Field

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It is shown that the discrete energy levels of an electron in a crystal are widened into narrow bands in strong magnetic fields. The structure of the energy zone in a crystal located in a magnetic field is studied. The possible influence of the broadening on the de Haas-van Alphen effect is discussed.

A FREE electron in a uniform magnetic field performs a finite motion[1] which corresponds to a classical revolution (at least in the direction perpendicular to the magnetic field H). The minimum quantum-mechanical "radius" of this revolution is

$$\alpha_0 = \sqrt{\hbar c / eH}. \qquad (1)$$

In addition to the magnetic field the electron is acted upon in the crystal by a periodic electric field (the lattice constant will be designated by a), and for all real fields H.

$$\varepsilon = a/\alpha_0 \ll 1 \qquad (2)$$

(thus, for example, with $H \sim 10^4$ oersteds and $a = 2.5 \times 10^{-8}$ cm we have $\epsilon = 10^{-2}$).

In the theory of electron motion in a crystal placed in a magnetic field the only terms that are retained (except in Ref. 2) are those which remain finite when $\epsilon \to 0$. As a result, the energy levels of the electron in a magnetic field are degenerate and depend, just as in the case of free electrons, on only two quantum numbers (see, for example, Refs. 3 and 4). If terms that vanish together with ϵ are retained the degeneracy is removed and the character of the spectrum is changed.

1. In the absence of the periodic field the energy levels of the motion of a free electron in the plane $\perp$ H are expressed by the equation $E_n = \mu H(n + 1/2)$ and the eigenfunctions are

$$\psi_{k,n} = e^{i k_1 x} \varphi_n\left(\frac{y + \alpha_0^2 k_1}{\alpha_0}\right),$$

where the φ_n are Chebyshev-Hermite functions. The energy is independent of the quantum number k_1 which determines the position of the "center of oscillation" of the electrons $y_0 = -\alpha_0^2 k_1$, since all y_0 are equivalent in free space. In a periodic field this equivalence disappears and the degeneracy is removed. In the approximation of weakly bound electrons (when the periodic field