Statistical Theory of the Atomic Nucleus. III

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The statistical model of the nucleus is investigated for uniform and nonuniform density distributions of nucleons. It is shown that a two nucleon central interaction potential, which contains the ordinary nonexchange repulsive force along with the spin exchange force, leads to a saturation of the nuclear binding energy and nuclear density.

THE theoretical explanation of the saturation of nuclear in complicated nuclei (saturation of the nuclear density and the binding energy) is a basic problem of nuclear physics. Up to now this problem has not received a satisfactory solution because of the fact that a precise law of interaction for nucleons has not been established. The saturation properties of the forces puts a definite limit on the choice of interaction between two nucleons. For example, the possibility of explanation of the stability of a nucleus with the aid of a single Wigner force of attraction (in the form of a rectangular well and Yukawa potential[1,2]) is excluded. Some spin forces $\sim (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$ also cannot guarantee stability of the nucleus.

In Refs. 3 and 4* it was shown that with the use of the statistical model a two nucleon central interaction potential, which contains the ordinary nonexchange repulsive force along with the spin force, permits us to obtain the saturation of the binding energy for normal density of nucleons which corresponds to the equilibrium nuclear radius $R_s = 1.48 \times 10^{-13} A^{1/3}$ cm. In this particular variant of the forces the general potential energy of the nucleus consists of the usual positive potential energy of repulsion V^0 and the negative exchange potential energy of attraction V^a of the nucleons. In Refs. 5 and 6, which are also devoted to the statistical calculation of the energy of the nucleus, and which take into account only the exchange forces of attraction of a different type of n-p and p-p interaction (with the force law and density functions different from those used in I, II) and excluding from consideration the potential energy V^0, that is determined by the usual density $\rho(r, r)$ of the nucleons.

In recently published works[7,8], the problem has also been investigated of the saturation of nuclear forces in complicated nuclei. It is pointed out in Ref. 7 that the repulsive forces between three nucleons, together with two nucleon forces in the pseudoscalar meson theory, give a satisfactory, qualitative explanation of nuclear saturation. In this case it is found that the saturation of the binding energy is approached for the equilibrium value of the nuclear radius $R_s = 1.61 \times 10^{-13} A^{1/3}$ cm. In Ref. 8 the authors show that consideration of the weak repulsion in the odd state (P state), in addition to the repulsive core (r_c) in the two nucleon potential of the pseudoscalar meson thoery permits us to obtain the saturation of the binding energy for the value of the nuclear radius. $R_s = 1.15 \times 10^{-13}/A^{1/3}$ cm without inclusion of the three nucleon repulsive forces. In these researches, as well as in Refs. 3 and 4, the statistical model of the nucleus with uniform density distribution of nucleons was employed.

Data on nuclear scattering of nucleons and electrons of high energy[9-14], on the shell structure of nuclei[15-19] and on x-ray emission of meson atoms[20,21] testify that a uniform density distribution of nucleons (and the Coulomb charge) in the nucleus cannot explain a series of experimental factors and force us to consider the possibility of use of a nuclear model based on the representation of a nonequilibrium distribution of particles (charges) inside the nucleus. In this connection, the investigation of the statistical model of the nucleus from the point of view of nonequilibrium density of nucleons is of considerable interest; it would permit better agreement of the ground state of the nucleus with the data on the shell structure of nuclei (empirical values of the mean square of the orbital momentum of nucleons, etc.).

The present paper is a development of our previous researches[3,4,22] for the case in which the density of the nucleons is characterized by a constant internal part (for $R < R_0$), and falls off on the boundary (for $r > R_0$) in exponential fashion. The role of the usual energy of repulsion of the nucleons is made clear in the investigation of

* We denote subsequent references to these works by I and II.

the problem of the saturation of forces in complicated nuclei. The energy of the nucleus is calculated as a function of the parameters of the nonuniform density distribution of the nucleons. It is shown that the saturation of the binding energy and the density of nucleons in the central portion of the nucleus is brought about by the presence of a repulsive force in the two nucleon potential and the Coulomb energy of the protons.

1. CALCULATION OF THE EXCHANGE AND ORDINARY POTENTIAL ENERGIES OF THE NUCLEUS

We assume that the interaction energy between two protons u_{12}^{pp}, between two neutrons u_{12}^{nn} and a proton and a neutron u_{12}^{np} is represented by the expression [see Eq. (1) in I]:

$$u_{12}^{pp} = u_{12}^{nn} = (g^2 + f^2(\vec{\sigma}_1\vec{\sigma}_2))\frac{e^{-k_0 r}}{r}; \quad (1)$$

$$u_{12}^{np} = (g^2 + f^2(\vec{\sigma}_1\vec{\sigma}_2))\frac{e^{-k_0 r}}{r} - (g^2 + f^2(\vec{\sigma}_1\vec{\sigma}_2))\frac{e^{-k_0 r}}{r} Q_H, \quad (2)$$

where Q_H is the Heisenberg operator for the permutation of the space and spin coordinates of particles 1 and 2; g^2 and f^2 are constants of the interaction; $r = |\mathbf{r}_1 - \mathbf{r}_2|$ is the distance between the nucleons 1 and 2; $1/k_0$ is a certain constant determining the radius of action of the nuclear forces. For simplicity, in first approximation, we omit the Coulomb repulsion between the protons. The wave functions for the protons $\psi_k(\mathbf{r}, s)$ and for the neutrons $\varphi_k(\mathbf{r}, s)$ in the nonrelativistic approximation can be represented in the form of a product of coordinate and spin functions:

$$\psi_k(\mathbf{r}, s) = \psi_k(\mathbf{r})\, a_k(s); \quad \varphi_k(\mathbf{r}, s) = \varphi_k(\mathbf{r})\, b_k(s) \quad (3)$$

(k = propagation vector, s = nucleon spin). For the isolated proton and neutron, we choose the coordinated part of the wave function in the form of a plane wave

$$\psi_k(r) = V^{-1/2}\exp\{i\mathbf{k}_p\mathbf{r}\}; \quad \varphi_k(\mathbf{r}) = V^{-1/2}\exp\{i\mathbf{k}_n\mathbf{r}\}, \quad (4)$$

where $\mathbf{k}_p$ and $\mathbf{k}_n$ are the propagation vectors of the proton and the neutron; V is the volume of the nucleus. For the spin functions $a_k(s)$ and $b_k(s)$ we have the Eqs. (6) of I.

Applying the Hartree-Fock method and considering Eqs. (1) and (2) for the total energy E of a nucleus consisting of Z protons and N neutrons ($A = N + Z$ = mass number of the nucleus), we obtain

$$E = V_{pp}^0 + V_{nn}^0 + V_{pn}^0 + V_{pp}^a + V_{nn}^a + V_{pn}^a + T_k^p + T_k^n, \quad (5)$$

where (the index i denotes either p or n everywhere):

$$V_{ii}^0 = \frac{1}{2} g^2 \iint (d\mathbf{r}_1)(d\mathbf{r}_2)\frac{e^{-k_0 r}}{r}\rho_i(\mathbf{r}_1)\rho_i(\mathbf{r}_2); \quad (6)$$

$$V_{pn}^0 = g^2 \iint (d\mathbf{r}_1)(d\mathbf{r}_2)\frac{e^{-k_0 r}}{r}\rho_p(\mathbf{r}_1)\rho_n(\mathbf{r}_2); \quad (7)$$

$$V_{ii}^a = -\frac{1}{4}(g^2 + 3f^2)\iint (d\mathbf{r}_1)(d\mathbf{r}_2)\frac{e^{-k_0 r}}{r}|\rho_i(\mathbf{r}_1, \mathbf{r}_2)|^2; \quad (8)$$

$$V_{pn}^a = -\frac{1}{2}(g^2 + 3f^2)\iint (d\mathbf{r}_1) \times (d\mathbf{r}_2)\frac{e^{-k_0 r}}{r}\rho_p(\mathbf{r}_1, \mathbf{r}_2)\rho_n^*(\mathbf{r}_1, \mathbf{r}_2); \quad (9)$$

$$T_k^p = \frac{h^2}{8\pi^2 M}\sum_{k=1}^{Z}\int \nabla\psi_k(\mathbf{r})\nabla\psi_k^*(\mathbf{r})(d\mathbf{r}_k), \quad T_k^n = \frac{h^2}{8\pi^2 M}\sum_{k=1}^{N}\int \nabla\varphi_k(\mathbf{r})\nabla\varphi_k^*(\mathbf{r})(d\mathbf{r}_k). \quad (10)$$

Here V_{pp}^0, V_{nn}^0 and V_{pn}^0 are the ordinary parts of the total potential energy of interaction of the protons with protons, neutrons with neutrons and protons with neutrons: V_{pp}^a, V_{nn}^a and V_{pn}^a are the exchange parts of these same energies; T_k^p and T_k^n are the kinetic energies of the protons and neutrons. The ordinary and mixed densities of protons and neutrons in Eqs. (6)-(9) are determined by the expressions

$$\rho_p(\mathbf{r}_1) = \rho_p(\mathbf{r}_1, \mathbf{r}_1) = \sum_{k=1}^{Z}\psi_k^*(\mathbf{r}_1)\psi_k(\mathbf{r}_1); \quad \rho_p(\mathbf{r}_1, \mathbf{r}_2) = \sum_{k=1}^{Z}\psi_k^*(\mathbf{r}_1)\psi_k(\mathbf{r}_2) \quad (11)$$

and similarly for ρ_n with φ in place of ψ and summation over N.

With the help of Eq. (4) we obtain from Eq. (11) the following relation between the densities and the maximum wave numbers of the protons K_p and of the neutrons K_n:

$$K_i^3 = 3\pi^2 \rho_i(r), \tag{12}$$

$$\rho_i(\mathbf{r}_1, \mathbf{r}_2) = (\sin K_i r - K_i r \cos K_i r)/\pi^2 r^3. \tag{13}$$

In this case we consider that each proton and neutron state is occupied by two protons and two neutrons, respectively, with antiparallel spins. Substituting the values (13) and (12) in Eqs. (8) and (9) and integrating, we obtain the exchange potential energies of the nucleons as functions of the densities $\rho_p(r)$, $\rho_n(r)$ in the form

$$V_{ii}^a = -(g^2 + 3f^2)(k_0^4/24\pi^3)\, V v(\rho_i), \tag{14}$$

$$V_{pn}^a = -(g^2 + 3f^2)(k_0^4/12\pi^3)\, V v(\rho_p, \rho_n), \tag{15}$$

where

$$v(\rho_i) = 6\alpha^4 \rho_i^{4/3} - \alpha^2 \rho_i^{2/3} - 8\alpha^3 \rho_i \operatorname{arctg} 2\alpha\rho_i^{1/3} + {}^1/_4 (1 + 12\alpha^2 \rho_i^{2/3}) \ln (1 + 4\alpha^2 \rho_i^{2/3}); \tag{16}$$

$$\begin{aligned} v(\rho_p, \rho_n) = {} & 3\alpha^4 (\rho_n \rho_p^{1/3} + \rho_n^{1/3} \rho_p) \\ & - \alpha^2 \rho_n^{1/3} \rho_p^{1/3} + {}^1/_4 [1 + 6\alpha^2 (\rho_n^{2/3} + \rho_p^{2/3}) \\ & - 3\alpha^4 (\rho_n^{2/3} - \rho_p^{2/3})^2] \ln \frac{1 + \alpha^2 (\rho_n^{1/3} + \rho_p^{1/3})^2}{1 + \alpha^2 (\rho_n^{1/3} - \rho_p^{1/3})^2} \\ & + 4\alpha^3 (\rho_n - \rho_p) \operatorname{arctg} \alpha (\rho_n^{1/3} - \rho_p^{1/3}) \\ & - 4\alpha^3 (\rho_n + \rho_p) \operatorname{arctg} \alpha (\rho_n^{1/3} + \rho_p^{1/3}). \end{aligned} \tag{17}$$

Here $V = 4\pi/3R^3$ is the volume of the nucleus; $\alpha = (3\pi^2)^{1/3} k_0$.

Equations (14) and (15) are the exchange energies of the nucleons of the nucleus with constant density distributions of protons ρ_p and neutrons ρ_n. For nonuniform densities, Eqs. (14) and (15) change so that multiplication by V is replaced by integration over the nuclear volume. Here ρ_p and ρ_n are the densities of protons and neutrons in the volume element $d\tau$ of the nucleus, so normalized that

$$\int \rho_p d\tau = Z; \quad \int \rho_n d\tau = N. \tag{18}$$

For the ordinary potential energy of repulsion of the nucleons of the nucleus (in the case of constant densities ρ_p and ρ_n), we get from Eqs. (6) and (7) [see the derivation of Eq. (14) in II]

$$V_{ii}^0 = \frac{2\pi g^2}{k_0^2} \rho_i^2(r) V; \qquad V_{pn}^0 = \frac{4\pi g^2}{k_0^2} \rho_p(r) \rho_n(r) V. \tag{19}$$

The total potential energy of the nucleus is composed of the total exchange potential energy of attraction and the total ordinary potential energy of repulsion and, in accord with Eqs. (14), (15) and (19), is equal to

$$\begin{aligned} W(\rho_n, \rho_p) = {} & (V_{pp}^a + V_{nn}^a + V_{pn}^a) \\ & + (V_{pp}^0 + V_{nn}^0 + V_{pn}^0) \\ = {} & -\frac{(g^2 + 3f^2) k_0^4}{24\pi^3} [v(\rho_p) + v(\rho_n) \\ & + 2v(\rho_p, \rho_n)] V + \frac{2\pi g^2}{k_0^2} [\rho_p(r) + \rho_n(r)]^2 V \end{aligned} \tag{20}$$

for the case of constant density of nucleons.

Taking into account Eqs. (4) and (12), we get from Eq. (10) the well-known expression for the kinetic energy of protons and neutrons:

$$T_k^i = (4\pi h^2/5M)(3\rho_i/8\pi)^{5/3} V; \tag{21}$$

M is the mean value of the masses of neutron and proton. For variable densities, multiplication by V in all formulas is replaced by integration over the volume. Thus we have obtained the exchange and ordinary parts of the total potential energy of the nucleus as a function of the density distribution of the particles and the parameters of the nuclear forces.

2. CASE OF A STATISTICAL NUCLEUS WITH $\rho_p(r) = \rho_n(r)$.

DETERMINATION OF THE PARAMETERS OF THE TWO NUCLEON POTENTIAL

Let us consider a simple case, assuming the density of protons and neutrons to be equal: $\rho_n(r) = \rho_p(r) = \frac{1}{2}\rho(r)$, where $\rho(r)$ is the total density of nucleons in the nucleus. Then the condition (18) takes the form

$$\int \rho(r)\, d\tau = A. \tag{22}$$

In accordance with Eq. (20), the total potential energy of the nucleus as a function of the density $\rho(r)$ is equal to (we compute the formulas for constant ρ):

$$W(\rho) = V^a_{ii,\,pn}(\rho) + V^0_{ii,\,pn} = \tag{23}$$

$$-\frac{g^2+3f^2}{6\pi^3}k_0^4 V v_A(\rho) + \frac{2\pi g^2}{k_0^2}\rho^2(r)V.$$

Here,

$$v_A(\rho) = 6\alpha_1^4\rho^{4/3} - \alpha_1^2\rho^{2/3} - 8\alpha_1^3\rho \operatorname{arctg} 2\alpha_1\rho^{1/3} \tag{24}$$

$$+\frac{1}{4}(1+12\alpha_1^2\rho^{2/3})\ln(1+4\alpha_1^2\rho^{2/3}),$$

$$\alpha_1 = (3\pi^2/2)^{1/3}/k_0.$$

With the help of Eq. (21) we will have for the Fermi kinetic energy of the nucleons:

$$T_k = T_k^p + T_k^n = (8\pi h^2/5M)(3\rho/16\pi)^{5/3}V. \tag{25}$$

For the determination of the parameters of the interaction potential (1) and (2), we limit ourselves to a consideration of the zeroth approximation, when we can consider the density $\rho(r)$ of the nucleons to be a constant in the volume V of the nucleus. It follows from Eq. (22) that $\rho = A/V = 3A/4\pi R^3$. Here R is the radius of the nucleus with constant particle density. Then, carrying out the substitution (18b) of II, we get the total energy E as a function of the radius R from Eqs. (23) and (25):

$$E = T_k(x) + V^0(x) + V^a_{(x)} \tag{26}$$

$$= \left\{\frac{a_2}{x^2} + \frac{a_3}{x^3} - a_1\Phi(\beta x)\right\}A,$$

where

$$a_1 = \frac{(9\pi)^{4/3}}{12\pi^2}\frac{\beta}{r_0}(g^2+3f^2); \tag{27}$$

$$a_2 = \left(\frac{3}{\pi}\right)^{4/3}\frac{3h^2}{160\,Mr_0^2}; \quad a_3 = \frac{3}{2}\frac{g^2}{\beta^2 r_0};$$

$$\Phi(\beta x) = \frac{1}{\beta x} - \frac{1}{6}\left(\frac{8}{9\pi}\right)^{2/3}\beta x$$

$$-\frac{4}{3}\left(\frac{8}{9\pi}\right)^{1/3}\operatorname{arctg} 2\left(\frac{9\pi}{8}\right)^{1/3}\frac{1}{\beta x}$$

$$+\left[\frac{1}{2}\left(\frac{8}{9\pi}\right)^{2/3}\beta x + \frac{1}{4}\left(\frac{8}{9\pi}\right)^{4/3}\beta^3 x^3\right]$$

$$\times \ln\left[1 + 4\left(\frac{9\pi}{8}\right)^{2/3}\frac{1}{\beta^2 x^2}\right];$$

$$\beta = k_0 r_0; \quad x = R/r_0 A^{1/3}. \tag{28}$$

The quantity x determines the departure of the density ρ from its equilibrium value. For $x = 1$ we have the normal density, for which the energy ought to have a minimum, equal to the empirical value. With the help of Eq. (20) of II, which determines the stable state of the system of nucleons, and the empirical value of the binding energy

$$\{E(x)\}_{x=1} = -\alpha_0 A; \quad \alpha_0 \approx 14 \text{ mev} \tag{29}$$

we find g^2 and f^2 from Eq. (26) as functions of the quantities β and r_0:

$$g^2 = -\frac{\alpha_0\Phi'(\beta) + a_2[2\Phi(\beta)+\Phi'(\beta)]}{m_0[3\Phi(\beta)+\Phi'(\beta)]}; \tag{30}$$

$$f^2 = \frac{m_0 - n_0\Phi(\beta)}{3n_0\Phi(\beta)}g^2 + \frac{a_2+\alpha_0}{3n_0\Phi(\beta)}, \tag{31}$$

where

$$m_0 = 3/2\,\beta^2 r_0; \quad n_0 = (3/\pi)^{2/3}3\beta/4r_0.$$

$\Phi(\beta)$ and $\Phi'(\beta)$, $\Phi''(\beta)$ are the values of the function $\Phi(\beta x)$ and its derivatives for $x = 1$. The inequality which determines the upper limit of the value of the constant $\beta = k_0 r_0$ has the form:

$$6a_2 - 12\frac{(\alpha_0+a_2)\Phi'(\beta) + 2a_2\Phi(\beta)}{3\Phi(\beta)+\Phi'(\beta)} \tag{32}$$

$$> \frac{3\alpha_0 + a_2}{3\Phi(\beta)+\Phi'(\beta)}\Phi''(\beta).$$

For the determination of the upper limit of the value of β we make use of two different empirical values of the parameter r_0, for judgment on the accuracy of which we make reference to in the literature. The data on nuclear scattering of high energy electrons, on the isotopic shift, on mesoatoms lead to a value of the radius R of a nucleus with constant density, equal to [10,11,20,21,23]:

$$r_0 = RA^{-1/3} \approx 1.2\times10^{-13}\text{ cm}. \tag{33}$$

On the other hand, it follows from the theory of α-decay, and the results of experiments on the scattering of fast neutrons, that [5,24,25]

$$r_0 = RA^{-1/3} \approx 1.48\times10^{-13}\text{cm and } 1.5\times10^{-13}\text{ cm}. \tag{34}$$

For the value of r_0 from Eq. (33), we find from Eq. (33) that $\beta < 1.80$. We then obtain a limit for the value of the effective radius of action of the nuclear forces:

$$1/k_0 > 0.667 \times 10^{-13} \text{ cm}. \quad (35)$$

For the value of Eq. (34), it follows from Eq. (32) that equilibrium of the nucleus is possible if $\beta < 1.75$. We then obtain

$$1/k_0 > 0.846 \times 10^{-13} \text{ cm}. \quad (36)$$

It follows from Eqs. (30)-(32) that the usual potential energy of repulsion $V^0 \sim \alpha_3$ of the nucleons plays an essential role in the problem under consideration, since this energy determines the value of the limiting effective radius of action of nuclear forces. In accord with Eqs. (35) and (36) we take the effective radius of action of nuclear forces $1/k_0$ to be equal to the Compton wavelength of the π-meson:

$$1/k_0 = \hbar/m_\pi c = 1.4 \times 10^{-13} \text{ cm} \quad (m_\pi = 276\, m_e). \quad (37)$$

Substituting the values of Eqs. (33), (34) and (37) into Eqs. (30) and (31) we find

$$g^2/\hbar c = 0.051; \quad f^2/\hbar c = 0.340 \quad (38)$$

$$\text{for} \quad r_0 = 1.2 \times 10^{-13} \text{ cm},$$

$$g^2/\hbar c = 0.132; \quad f^2/\hbar c = 0.479 \quad (39)$$

$$\text{for} \quad r_0 = 1.48 \times 10^{-13} \text{ cm}.$$

For the values of Eq. (38), the usual energy of repulsion, the exchange energy of attraction and the total potential energy per nucleon are, respectively (for $x = 1$),

$$V^0/A = 17 \text{ mev}; \quad V^a/A = -51 \text{ mev}; \quad (40)$$

$$T/A = 20 \text{ mev};$$

$$W/A = V^0/A + V^a/A = -34 \text{ mev},$$

and for (39) we have

$$V^0/A = 23.6 \text{ mev}; \quad V^a/A = -50.8 \text{ mev}; \quad (41)$$

$$T/A = 13.2 \text{ mev};$$

$$W/A = -27.2 \text{ mev}.$$

The dependencies of all quantities on x, computed from Eq. (26), are shown graphically in Figs. 1 and 2. It is evident from these graphs that the total potential energy per nucleon has a minimum, equal to

$$W/A = -43.67 \text{ mev}$$

$$\text{for } R_m = 0.7\, r_0 A^{1/3} = 0.840 \cdot 10^{-13} A^{1/3}$$

for the case (38) and

$$W/A = -30.54 \text{ mev}$$

$$\text{for } R_m = 0.8\, r_0 A^{1/3} = 1.184 \times 10^{-13} A^{1/3} \text{ cm}$$

for case (39). Consideration of the kinetic energy leads to the result that the minimum mean energy E/A, equal to the empirical value ~ -14 mev, increases for $x = 1$, i.e., at the equilibrium, value of the nuclear radius R, it is somewhat larger

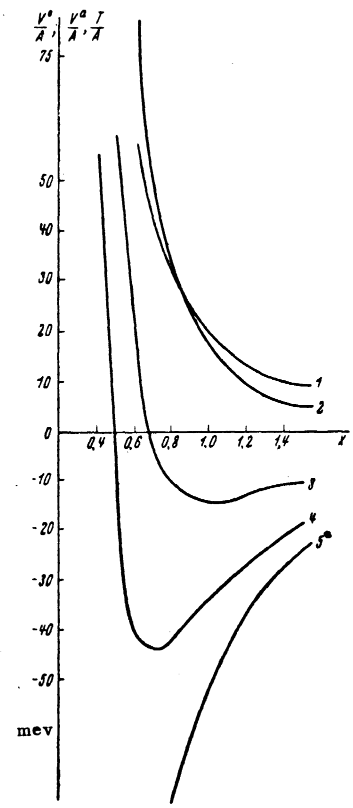

FIG. 1. Potential, kinetic and total energies per nucleon as functions of $x = R/r_0 A^{1/3}$ for values of the parameters from Eq. (38): 1 - T/A, 2 - V^0/A, 3 - E/A, 4 - $(V^0/A + V^a/A)$, 5 - V^a/A.

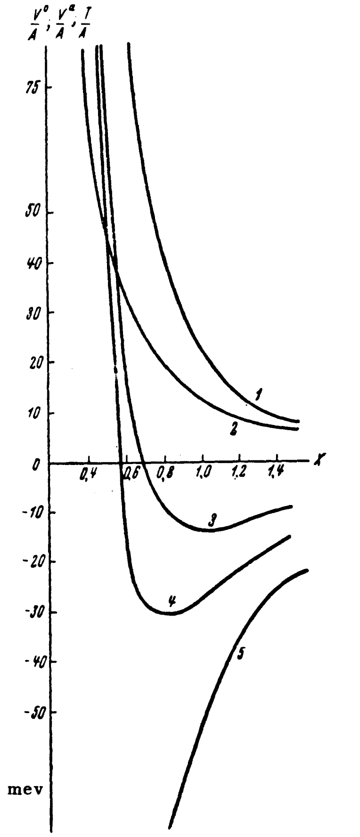

FIG. 2. Potential, kinetic and total energies per nucleon as functions of x for values of the parameters from Eq. (39). 1-V^0/A, 2-T/A, 3-E/A, 4-($V^0/A + V^a/A$), 5-V^a/A.

than R_m. Thus, in the simplest case of the statistical model (constant density of particles), a two nucleon potential which contains both the exchange forces and the usual short range forces of repulsion, leads to a saturation of the binding energy which sets in for normal density ($x = 1$).

3. CALCULATION OF THE BINDING ENERGY OF THE NUCLEUS WITH NONUNIFORM DENSITY DISTRIBUTION OF NUCLEONS

Taking into account the empirical facts that the mean binding energy per nucleon is almost constant and the nuclear volume is proportional to A, the density $\rho(r)$ of particle distribution in the nucleus can be taken equal to[22] (see also Refs. 26-28)

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r \leqslant R_0, \\ \rho_0 \cdot \exp\{-(r - R_0/a)\} & \text{for } r > R_0. \end{cases} \quad (42)$$

Here ρ_0 is the particle density (independent of r) in the central part of the nucleus; R_0 is the radius of the central part, a is the thickness of the surface layer, where the density falls off exponentially. For such a density choice, the nucleus has a dense interior part (core) and a diffuse periphery. More recently, Jastrow and Roberts[26] chose, for the explanation of the nuclear scattering of neutrons in the energy range 14-270 mev, a density of nuclear matter of the same for as in the work of one of us[22] [see Eq. (42)]. We note that the theoretical analysis of experimental data on nuclear scattering of nucleons and electrons of high energy shows that the nuclear model (42) gives better agreement with experiment than the model of Born and Yang[27,28].

We proceed to the calculation of the binding energy of the nucleus as a function of the variation parameters a, R_0 and ρ_0, with the help of the Ritz method. Carrying out integration over the volume of the nucleus, we obtain for the exchange and ordinary potential energies

$$V^a = -\frac{2}{9}(g^2 + 3f^2)\frac{k_0^4 R_0^3}{\pi^2}\{B_0(q) + \varepsilon_0 B_1(q) + \varepsilon_0^2 B_2(q) + \varepsilon_0^3 B_3(q)\}, \quad (43)$$

$$V^0 = \frac{g^2 k_0^4}{54\pi^2} q^6 R_0^3 \left\{1 + \frac{3}{2}\varepsilon_0 + \frac{3}{2}\varepsilon_0^2 + \frac{3}{4}\varepsilon_0^3\right\}, \quad (44)$$

where

$$B_0(q) = \frac{3}{8}q^4 - \frac{1}{4}q^2 + \frac{1}{4}(1 + 3q^2)\ln(1 + q^2) - q^3 \operatorname{arctg} q; \quad (45)$$

$$B_1(q) = \frac{27}{32}q^4 - 3q^2 + \frac{15}{8}\ln(1 + q^2) + \frac{27}{8}q^2\ln(1+q^2) - 3q^3\operatorname{arctg} q + \frac{9}{4}X_1(q);$$

$$B_2(q) = \frac{81}{64}q^4 - \frac{129}{8}q^2 + \frac{57}{8}\ln(1 + q^2) + \frac{81}{8}q^2\ln(1 + q^2) - 6q^3\operatorname{arctg} q + \frac{45}{4}X_1(q) + \frac{27}{2}X_2(q);$$

$$B_3(q) = \frac{243}{256}q^4 - \frac{291}{8}q^2 + \frac{195}{16}\ln(1 + q^2) + \frac{243}{16}q^2\ln(1 + q^2) - 6q^3\operatorname{arctg} q + \frac{171}{8}X_1(q) + \frac{135}{4}X_2(q) + \frac{81}{4}X_3(q);$$

$$X_1(q)=\int_0^\infty \ln(1+q^2e^{-2x})\,dx;$$

$$X_2(q)=\int_0^\infty x\ln(1+q^2e^{-2x})\,dx;$$

$$X_3(q)=\int_0^\infty x^2\ln(1+q^2e^{-2x})\,dx;$$

$$\varepsilon_0=a/R_0;\quad q=(12\pi^2)^{1/3}\rho_0^{1/3}/k_0.$$

For the kinetic energy we get

$$T_k=\frac{h^2k_0^5q^5}{480\pi^3M}R_0^3\left\{1+\frac{9}{5}\varepsilon_0+\frac{54}{25}\varepsilon_0^2+\frac{162}{125}\varepsilon_0^3\right\}. \quad (46)$$

Making use of Eq. (45), we get from Eqs. (22) and (42) a single relation among the three parameters R_0, q and ϵ_0:

$$R_0=(9\pi)^{1/3}(k_0q)^{-1}A^{1/3}(1+3\varepsilon_0+6\varepsilon_0^2+6\varepsilon_0^3)^{-1/3}. \quad (47)$$

Eliminating R_0 from Eqs. (43), (44) and (46), we reduce the total energy of the nucleus to the following form, making use of Eq. (47):

$$E=V^a+V^0+T_k \quad (48)$$

$$=\frac{A}{1+3\varepsilon_0+6\varepsilon_0^2+6\varepsilon_0^3}\Big\{-b_0\left[\frac{B_0(q)}{q^3}+\varepsilon_0\frac{B_1(q)}{q^3}+\varepsilon_0^2\frac{B_2(q)}{q^3}+\varepsilon_0^3\frac{B_3(q)}{q^3}\right]$$

$$+b_1q^3[1+{}^3/_2\,\varepsilon_0+{}^3/_2\,\varepsilon_0^2+{}^3/_4\,\varepsilon_0^3]+b_2q^2\left[1+{}^9/_5\,\varepsilon_0+\frac{54}{25}\varepsilon_0^2+\frac{162}{125}\varepsilon_0^3\right]\Big\}.$$

Here,

$$b_0=2k_0(g^2+3f^2)/\pi;\quad b_1=g^2k_0^2/6\pi;\quad b_2=3h^2k_0^2/160\,\pi^2M.$$

The energy E depends on the two variation parameters g and ϵ_0. The parameter q characterizes the density ρ_0 of the central part of the nucleus, and ϵ_0 gives the ratio of the region of exponential decay to the linear dimension of the central part of the nucleus (which has constant density). From the condition of minimum energy $\partial E/\partial q=0$, we find the equation which connects the parameters q and ϵ_0:

$$\Phi_1(q,\varepsilon_0)=-b_0[C_0(q)+\varepsilon_0C_1(q)+\varepsilon_0^2C_2(q)+\varepsilon_0^3C_3(q)]+3b_1q^2(1+{}^3/_2\,\varepsilon_0+{}^3/_2\,\varepsilon_0^2+{}^3/_4\,\varepsilon_0^3)+2b_2q\left(1+\frac{9}{5}\varepsilon_0+\frac{54}{25}\varepsilon_0^2+\frac{162}{125}\varepsilon_0^3\right)=0, \quad (49)$$

where

$$C_j(q)=\frac{\partial}{\partial q}\left(\frac{B_j(q)}{q^3}\right);\quad j=0,1,2,3$$

$$X_1'(q)=\frac{\ln(1+q^2)}{q};\quad X_2'(q)=\frac{X_1(q)}{q};\quad X_3'(q)=2\frac{X_2(q)}{q}.$$

For each chosen value of q in Eq. (49), we can find the corresponding ϵ_0 and thus establish the dependence of q on ϵ_0. For this purpose, we first determine the interval of possible values of the parameter q and consider two limiting cases of the density distribution (42). In the limiting case

$$R_0=0\quad\text{or}\quad\varepsilon_0\to\infty \quad (50)$$

Eq. (42) goes over to a density that falls off according to the simple exponential

$$\rho(r)=\rho_0\exp(-r/a). \quad (51)$$

With the aid of Eqs. (22) and (45), we find the dependence between the parameters q and a:

$$a=(3\pi/2)^{1/3}A^{1/3}/k_0q. \quad (52)$$

Setting $R=0$ in the general expression for the energy (48) (or $\epsilon_0\to\infty$) and considering Eq. (52), we get the binding energy E of the nucleus with density distribution (51) in the form

$$E=\left\{-\frac{b_0}{6}\frac{B_3(q)}{q^3}+\frac{b_1}{8}q^3+\frac{27}{125}b_2q^2\right\}A. \quad (53)$$

We get the equation defining q from the condition of minimum energy:

$$\Phi_2(q)=-\frac{b_0}{6}\frac{\partial}{\partial q}\left(\frac{B_3(q)}{q^3}\right)+\frac{3}{8}b_1q^2+\frac{54}{125}b_2q=0. \quad (54)$$

For the values (38) of the constants q^2, f^2, we find from Eq. (54) that $q = 6.2$ or, from Eq. (52),

$$a = 0.39\times 10^{-13} A^{1/3}\ \text{cm} \tag{55}$$

for the values of Eq. (39), we get $q = 4.7$ from Eq. (54), or

$$a = 0.49\times 10^{-13} A^{1/3}\ \text{cm}. \tag{56}$$

For these values of the parameter q or ρ_0, the mean binding energy per nucleon is, from Eq. (53), $E/A \approx -8.5$ mev.

It was shown in Ref. 13 that the charge density distribution in the nucleus of the type (51) gives better agreement with the experimental data on the nuclear scattering of fast electrons on beryllium, gold and lead (energy of the electrons $\sim$ 125 mev) and tantalum (energy $\sim$ 150 mev) for values of the parameter $a = 0.636 \times 10^{-13}$ cm for Be; $(2.3 \pm 0.3) \times 10^{-13}$ cm for Au; 2.36×10^{-13} cm for Pb and $(2.80 \pm 0.3) \times 10^{-13}$ cm for Ta. From our theoretical expressions (55) and (56) for a, we have, respectively, $a = 0.81 \times 10^{-13}$ cm for Be; 2.27×10^{-13} cm for Au; 2.31×10^{-13} cm for Pb, 2.21×10^{-13} cm for Ta and $a = 1.02 \times 10^{-13}$ cm for Be; 2.85×10^{-13} cm for Au; 2.90×10^{-13} cm for Pb; 285×10^{-13} cm for Ta.

In the other limiting case,

$$a = 0 \quad \text{or} \quad \varepsilon_0 = 0 \tag{57}$$

the density relation (42) gives a constant density for the entire extent of the nucleus:

$$\rho(r) = \begin{cases} \rho_0 & \text{for} \quad r \leqslant R_0, \\ 0 & \text{for} \quad r > R_0. \end{cases} \tag{58}$$

Taking into account Eqs. (45) and (57), we have the following relation between q and R_0 from (47):

$$R_0 = (9\pi)^{1/3} A^{1/3} / k_0 q. \tag{59}$$

Substituting Eqs. (57) and (59) in (43), (44) and (46), we obtain the binding energy of a nucleus with constant particle density in the form of a function of $q (\sim \rho_0)$:

$$E = \left\{ -b_0 \frac{B_0(q)}{q^3} + b_1 q^3 + b_2 q^2 \right\} A. \tag{60}$$

It follows from the condition $\partial E/\partial q = 0$ that: a) in the case (38), the binding energy per nucleon has a minimum with a value of ~ -14 mev for $q = 3.5$ or, in accord with Eq. (59), for $R_0 = 1.2 \times 10^{-13} A^{1/3}$ cm; b) for (39), the binding energy per nucleon has a minimum with value ~ -14 mev for the value $q = 2.8$ or for $R_0 = 1.48 \times 10^{-13} A^{1/3}$ cm. Thus, for ϵ_0, which changes from 0 to ∞, all the values of the parameter q, which are determined from the requirement of minimum energy, lie in the interval

$$3.5 \leq q \leq 6.2 \text{ for the value of (38)}, \tag{61}$$

$$2.8 \leq q \leq 4.7 \text{ for the value of (39)}. \tag{62}$$

The dependence of the parameter q on ϵ_0, computed from Eq. (49) in the case of Eqs. (61) and (62), is shown in Fig. 3, from which it is seen that in both cases the parameter q reaches its limiting value $q = 6.2$ and $q = 4.7$ for $\epsilon_0 \approx 13.798$.
As is evident, in the approximation under consideration, when we neglected the Coulomb and surface energies, the density of the nucleons in the central part of the nucleus ρ_0 and the ratio a/R_0 are constant quantities which do not depend on the mass number A.

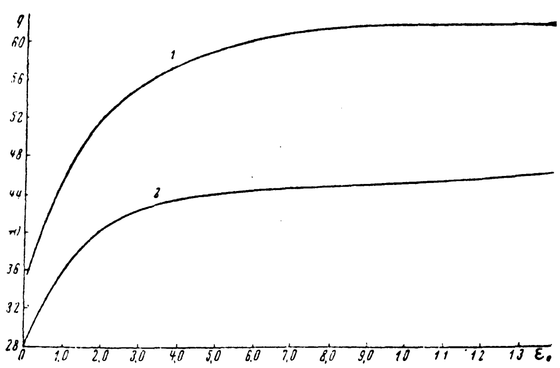

FIG. 3. Dependence of q upon ϵ_0. 1 – for values of the parameters from Eq. (38), 2 – for values of the parameters from Eq. (39).

According to Eqs. (45) and (47), the parameters R_0, a and $\bar{r}_0$ can be represented as functions of q and ϵ_0:

$$R_0 = R_0^*(q, \varepsilon_0) A^{1/3}; \quad a = a_0(q, \varepsilon_0) A^{1/3}; \tag{63}$$

$$\bar{r}_0 = R_0 + a = r_0^*(q, \varepsilon_0) A^{1/3},$$

where the following notation is introduced:

$$R_0^*(q, \varepsilon_0) = (9\pi)^{1/3}(k_0 q)^{-1}(1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)^{-1/3};$$

$$a_0(q, \varepsilon_0) = \varepsilon_0 R_0^*(q, \varepsilon_0);$$

$$r_0^*(q, \varepsilon_0) = R_0^*(q, \varepsilon_0) + a_0(q, \varepsilon_0).$$

For each pair of values of q and ϵ_0 we obtain from Fq. (63) the corresponding values of $R_0^*(q, \epsilon_0)$, $a_0(q, \epsilon_0)$ and $r_0^*(r, \epsilon_0)$. The results of the calculation are shown in Fig. 4, from which it follows that for a given A: a) the initial point of these curves with coordinates

$$R_0^*(q, \varepsilon_0) \approx 1.2\times 10^{-13} \text{ cm};$$

$$a_0(q, \varepsilon_0) = 0; \quad \rho_0 = 0.362\, k_0^3;$$

$$R_0^*(q, \varepsilon_0) \approx 1.48\times 10^{-13} \text{ cm};$$

$$a_0(q, \varepsilon_0) = 0; \quad \rho_0 = 0.185\, k_0^3$$

corresponds to a nuclear model with constant density of nucleons (58); b) the final point of the curves with coordinates

$$a_0(q, \varepsilon_0) \approx 0.39\times 10^{-13} \text{ cm};$$

$$R_0^*(q, \varepsilon_0) = 0; \quad \rho_0 = 2.012\, k_0^3;$$

$$a_0(q, \varepsilon_0) \approx 0.49\times 10^{-13} \text{ cm};$$

$$R_0^*(q, \varepsilon_0) = 0; \quad \rho_0 = 0.877\, k_0^3$$

corresponds to the nuclear model with variable density (51). The intervals $X_1(q)$, $X_2(q)$ and $X_3(q)$ were computed numerically.

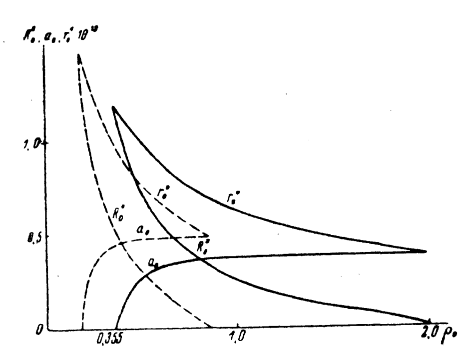

FIG. 4. Dependence of R_0^*, r_0^* and a_0 on ρ_0 (in units of k_0^3) for values of the parameters from Eq. (38) (Solid curves) and for values from Eq. (39) (broken curves).

4. CALCULATION OF THE ENERGY OF THE NUCLEUS AND THE DENSITY DISTRIBUTION OF NUCLEONS WITH CONSIDERATION OF COULOMB AND SURFACE EFFECTS

In Sec. 3 for the calculation of the density of nucleons in a nucleus, we took into account only the energy of the specific nuclear forces and neglected the Coulomb repulsion of the protons. Now we determine the parameters of the density (42), taking into account the Coulomb energy of the protons and the Weizsacker correction to the kinetic energy of the particles. In the Hartree-Fock approximation for the Coulomb energy of a nucleus with Z protons, we have

$$E_c = E_c^0 + E_c^a, \tag{64}$$

where

$$E_c^0 = \frac{1}{2}\iint (d\mathbf{r}_1)(d\mathbf{r}_2)\frac{e^2}{r}\rho_p(\mathbf{r}_1)\rho_p(\mathbf{r}_2), \tag{65}$$

$$E_c^a = -\frac{1}{4}\iint (d\mathbf{r}_1)(d\mathbf{r}_2)\frac{e^2}{r}|\rho_p(\mathbf{r}_1,\mathbf{r}_2)|^2. \tag{66}$$

The first term in (64) is the usual Coulomb energy, the second, the Coulomb exchange energy.

In the case of variable density $\rho_p(r)$, we get E_c^a from (66) with the help of Eqs. (12) and (13):

$$E_c^a = -\frac{3e^2}{4\pi}(3\pi^2)^{1/3}\int \rho_p^{4/3}(r)\,d\tau. \tag{67}$$

Substituting for ρ_p in Eqs. (65) and (67) the expression (42), and carrying out the integration, we find, in the notation of (45),

$$E_c^0 = \frac{e^2 k_0^6}{540\pi^2} R_0^5 q^6\left[1 + \frac{15}{2}\varepsilon_0^2 + \frac{75}{4}\varepsilon_0^3 + \frac{75}{4}\varepsilon_0^4 + \frac{75}{8}\varepsilon_0^5\right], \tag{68}$$

$$E_c^a = -\frac{e^2 k_0^4}{48\pi^2} R_0^3 q^4\left[1 + \frac{9}{4}\varepsilon_0 + \frac{27}{8}\varepsilon_0^2 + \frac{81}{32}\varepsilon_0^3\right]. \tag{69}$$

Making use of (47), we obtain the Coulomb energy of the nucleus as a function of the parameters ϵ_0 and q:

$$\begin{aligned} E_c &= E_c^0 + E_c^a \\ &= b_3 A^{5/3} q\left(1 + \frac{15}{2}\varepsilon_0^2 + \frac{75}{4}\varepsilon_0^3 + \frac{75}{4}\varepsilon_0^4 + \frac{75}{8}\varepsilon_0^5\right) \\ &\quad \times (1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)^{-5/3} \\ &\quad - b_4 A q\left(1 + \frac{9}{4}\varepsilon_0 + \frac{27}{8}\varepsilon_0^2 + \frac{81}{32}\varepsilon_0^3\right) \\ &\quad \times (1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)^{-1}. \end{aligned} \tag{70}$$

Here

$$b_3 = 9e^2k_0 / 60\,(9\pi)^{1/3}; \qquad b_4 = 3e^2k_0 / 16\pi.$$

The kinetic energy associated with a nonuniform density of nucleons on the periphery of the nucleus is determined by the Weizsacker correction, which in our case has the form:

$$T_{\rm B} = T_{\rm B}^p + T_{\rm B}^n = \frac{\hbar^2}{32\pi^2 M} \int \frac{(\operatorname{grad}\rho)^2}{\rho}\, d\tau. \qquad (71)$$

Keeping (42) and (47) in mind, we get from (71) after integration,

$$T_{\rm B} = b_5 A^{1/3} q^2 (2 + 2\varepsilon_0 + \varepsilon_0^{-1}) \times (1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)^{-1/3}, \qquad (72)$$

where

$$b_5 = \frac{(9\pi)^{1/3}}{96\pi^3} \frac{\hbar^2 k_0^2}{M}.$$

The total energy of the nucleus in this case will be equal to

$$E_n = E(q, \varepsilon_0) + E_c(q, \varepsilon_0) + T_B(q, \varepsilon_0) = E(q, \varepsilon_0) + \frac{A}{1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3} \times \Big\{ b_3 A^{2/3} q \frac{1 + {}^{15}/_2\varepsilon_0^2 + {}^{75}/_4\varepsilon_0^3 + {}^{75}/_4\varepsilon_0^4 + {}^{75}/_8\varepsilon_0^5}{(1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)^{2/3}} - b_4 q \left(1 + \frac{9}{4}\varepsilon_0 + \frac{27}{8}\varepsilon_0^2 + \frac{81}{32}\varepsilon_0^3\right) + b_5 q^2 A^{-2/3} (2 + 2\varepsilon_0 + \varepsilon_0^{-1}) \times (1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)^{2/3} \Big\}. \qquad (73)$$

Here $E(q, \epsilon_0)$ is the energy of the nucleus without consideration of the Coulomb and surface kinetic energy which is determined by Eq. (48).

The equation which relates q and ϵ_0,

$$\partial E_n / \partial q = F_1(q, \varepsilon_0, A) = 0, \qquad (74)$$

was solved by us for ϵ_0 with q in the intervals (61) and (62) for the values $A = 40, 60, 80, 120, 140, 180, 200$ and 220. For each pair of values (q, ϵ_0) found from Eq. (74), we calculated [by means of Eq. (73)] the corersponding value of E/A. The dependence of E/A on ϵ_0 for certain A is given in Fig. 5. It is evident from these graphs that to each A there corresponds only one pair of values of the parameters (q, ϵ_0) for which the nucleus possesses minimum energies almost equal to the empirical values; these pairs of values (q, ϵ_0) are shown in Tables I and II. There are also plotted in these tables the values of the parameters R_0, a, $\overline{r_0}$ and ρ_0 with the aid of Eqs. (63) and (45), in which cases the Coulomb and surface energies have been taken into account.

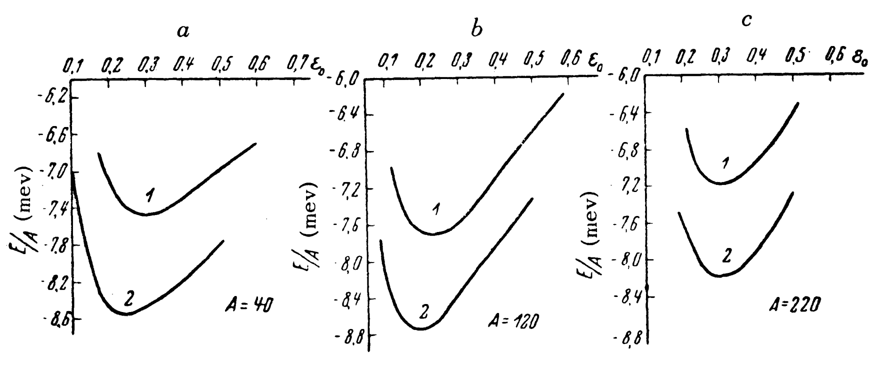

FIG. 5. Binding energy per nucleon E/A as functions of $\epsilon_0 = a/R_0$ for the values $A = 40, 120, 220$. *1* – for values of the parameters from Eq. (38), *2* – for values from Eq. (39).

TABLE I. For Values of the Parameters Used in Eq. (38)

A	q	$\varepsilon_0=\frac{a}{R_0}$	$R_0A^{-1/3}\times 10^{13}$	$a\cdot A^{-1/3}\times 10^{13}$	$\bar{r}_0A^{-1/3}\times 10^{13}$	ρ_0 (в единицах k_0^3)
40	3.7	0.300	0.872	0.262	1.134	0.427
60	3.69	0.265	0.908	0.241	1.149	0.424
80	3.65	0.242	0.941	0.228	1.169	0.410
120	3.64	0.218	0.968	0.211	1.179	0.407
140	3.64	0.210	0.976	0.205	1.181	0.407
180	3.63	0.205	0.984	0.202	1.186	0.404
200	3.63	0.205	0.984	0.202	1.186	0.404
220	3.63	0.205	0.984	0.202	1.186	0.404

TABLE II. For Values of the Parameters Used in Eq. (39)

A	q	$\varepsilon_0=\frac{a}{R_0}$	$R_0A^{-1/3}\times 10^{13}$	$aA^{-1/3}\times 10^{13}$	$\bar{r}_0A^{-1/3}\times 10^{13}$	ρ_0 (в единицах k_0^3)
40	3.0	0.233	1.155	0.269	1.424	0.228
60	3.0	0.215	1.178	0.253	1.431	0.228
80	3.0	0.204	1.192	0.243	1.435	0.228
120	3.0	0.195	1.204	0.235	1.439	0.228
140	3.0	0.195	1.204	0.235	1.439	0.228
180	3.0	0.195	1.204	0.235	1.439	0.228
200	3.0	0.195	1.204	0.235	1.439	0.228
220	3.0	0.195	1.204	0.235	1.439	0.228

It is evident from the tables and graphs that consideration of the Coulomb and surface kinetic energies leadsto the following results. First, for each A, the parameters (q, ϵ_0) of the nuclear density (42) have only one pair of values, for which equilibrium of the nucleus is possible; this shows that the existence, in the medium and heavy nuclei, of a core (of radius R_0) with almost constant density is due to the effect of the usual repulsive force in the two nucleon potential and of Coulombrepulsion, without which the density of the distribution would be close to (51) for all A (see Sec. 3).

Second, the ratio $\epsilon_0 = a/R_0$ is a definite function of the mass number A of the nucleus (see TablesI, II); for $A \geq 120$, the ratio a/R_0 is practically independent of A: $\epsilon_0 \approx 0.2$.

Third, the value of the parameter q is virtually independent of A for $A > 50$, i.e., for medium and heavy nuclei, the density of nucleons in the core of the nucleus is constant. This conclusion is in agreement with the known empirical law, according to which, for medium and heavy nuclei, the density of nucleons inside the nucleus does not depend on the mass number A. We give here the values of the parameters R_0, a, ρ_0 and $\bar{r}_0$ – the particle distribution densities (42) for nuclei with $A \geq 120$ for two systems of valuesof the constants (38) and (39):

$$R_0 = 0.98\times 10^{-13}A^{1/3}; \qquad a = 0.21\times 10^{-13}A^{1/3};$$
$$\rho_0 = 0.41k_0^3; \qquad \bar{r}_0 = 1.19\times 10^{-13}A^{1/3}; \tag{75}$$

$$R_0 = 1.20\times 10^{-13}A^{1/3}; \qquad a = 0.24\times 10^{-13}A^{1/3};$$
$$\rho_0 = 0.23k_0^3; \qquad \bar{r}_0 = 1.44\times 10^{-13}A^{1/3}. \tag{76}$$

The average energy per nucleon E/A, computed as a function of A, is shown graphically in Fig. 6. The dependence of E/A on A is in good agreement with the empirical curve of the binding energy, computed from the mass defect. Thus, for nuclei with $A \geq 120$, the parameters R_0 and a are obtained proportional to $a^{1/3}$. We note that a dependence of the form $a \sim A^{1/3}$, $R_0 \sim A^{1/3}$ follows from the connection between the structure of the nuclear shells and the density of the nucleons[16,17,27,28]. The density of $\rho(r)$ is shown in Fig. 7 as a function of r for various A

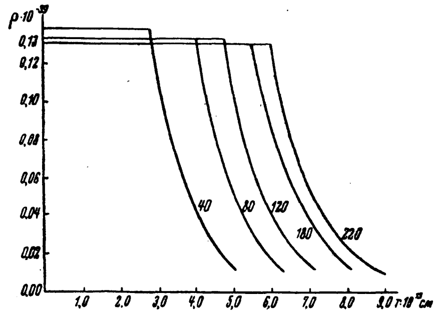

FIG. 6. Mean binding energy per nucleon E/A as a function of A for values of the parameters from Eq. (38). For values of the parameters from Eq. (39) for A=40-220, the quantity $E/A \approx -8.5$ mev.

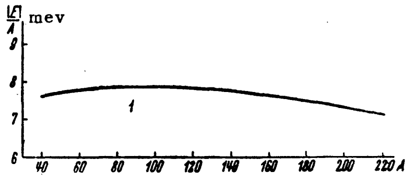

FIG. 7. Dependence of the density distribution $\rho(r)$ on r for $A = 40$, 80, 120, 180 and 220, and for values of the parameters from Eq. (39).

(see *Note added in proof* at end of paper).

In conclusion, we convey our deep gratitude to Prof. A. A. Sokolov for suggesting the theme and for discussions of the results. We also express our thanks to L. I. Morozovskii for help in the computations.

Note added in proof: We have the value

$$\sqrt{\langle r^2\rangle} \approx 1.1 \times 10^{-13} A^{1/3} \text{ cm}$$

for the mean square of the nuclear radius from the density function (42).

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184