with the usual linear interactions.

The method may also be generalized to stationary problems for which one cannot determine initial conditions. In that case the problem reduces to a set of homogeneous integral equations which may be obtained from (12) by omitting the terms in U^0 In solving stationary problems one may use the method of residues. Problems involving several particles lead to equations of the Bethe-Salpeter type.

In the region of low energies the method proposed above leads, as a rule, to the same results as perturbation theory (in corresponding approximations). At high energies, however, the corrections may be appreciable, in particular as regards the relative importance of various processes.

It is a valuable duty to thank I. M. Lomsadze for valuable discussions.

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Translated by R. E. Peierls
180

SOVIET PHYSICS, JETP VOLUME 3, NUMBER 5 DECEMBER, 1956

Theory of Wave Motion of an Electron Plasma

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(Submitted to JETP editor March 19, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 30, 915-928 (May, 1956)

A general investigation of the nonlinear wave motions of an electron plasma has been carried out for arbitrary electron velocities. Temperature effects are not taken into account, and the state of the plasma is characterized not by a distribution function, but by the particle density. A correspondence is established between the wave motion of the plasma and the motion of a nonrelativistic particle in a certain potential field. The variation of the frequency of longitudinal vibrations on the velocity amplitude has been determined. Nonlinear transverse vibrations of the plasma, and vibrations close to these, are also considered, and their frequencies determined. A number of relations are established for the complicated longitudinal transverse plasma oscillations.

1. FUNDAMENTAL EQUATIONS

IN the study of the oscillatory behavior of an electron plasma, i.e., of an electron gas neutralized by ions, or a neutralized electron beam, it is usually assumed that the electron velocities and the density fluctuations are small, so that one may use a linearized system of equations. This scheme makes it possible to determine the frequencies of oscillation and to discuss, by means of gas-kinetic considerations, the part played by temperature effects[1,2], which turn out in general to be unimportant.

Nonlinear plasma oscillations were considered in a previous paper[3], in which temperature effects were neglected, and the electron velocity was assumed to be finite, but essentially nonrelativistic. Under these assumptions it is found that the oscillation frequency is independent of the velocity amplitude, and obeys the classical formula of Langmuir.

The purpose of the present paper is to investigate the oscillatory motion of the plasma quite generally, for arbitrary velocities. But, as in Ref. 3, we shall neglect temperature effects, i.e, we shall assume the plasma temperature to be zero. This approximation is very natural when we are investigating nonlinear oscillations even in a "high-temperature" plasma, and even more so in the study of plasma oscillations in electron beams, where the temperature is practically zero. Under these conditions it is not necessary to introduce a distribution function to specify the state of the plasma, but one may use the electron density, which depends on the coordinates and time. Some of the conclusions derived below apply also to the case of a plasma in an external magnetic field.

Assuming the plasma infinitely extended, we shall start from the following fundamental equations for the quantities E, H, the density n and the electron velocity $\mathbf{v}$:

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \tag{1}$$

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} ne\,\mathbf{v}; \qquad \operatorname{div} \mathbf{H} = 0;$$

$$\operatorname{div} \mathbf{E} = 4\pi e (n - n_0);$$

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v}\nabla)\mathbf{p} = e\mathbf{E} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]$$

Here n_0 is the equilibrium density of electrons, equal also to the density of the ions, which are assumed infinitely heavy and immobile; p is the electron momentum, equal to $m\mathbf{v}[1-(v^2/c^2)]^{-1/2}$.

Our problem consists of the general investigation of the wave motions of the plasma, i.e., of such electron motions for which all the variables entering into Eq. (1) are functions not of r and t separately, but only of the combination $\mathbf{i}\cdot\mathbf{r} - Vt$, where i is a constant unit vector, and V a constant. The meaning of this type of solution is that it represents plane waves travelling in the direction i with speed V.

If we denote derivatives with respect to this argument by a prime, we can rewrite Eqs. (1) in the form

$$[\mathbf{i}\times\mathbf{E}'] = \beta\,\mathbf{H}', \tag{2}$$

$$[\mathbf{i}\times\mathbf{H}'] = -\beta\mathbf{E}' + \frac{4\pi}{c} en\,\mathbf{v}, \tag{3}$$

$$\mathbf{i}\mathbf{H}' = 0, \tag{4}$$

$$\mathbf{i}\mathbf{E}' = 4\pi e (n - n_0), \tag{5}$$

$$(\mathbf{i}\mathbf{v} - V)\,\mathbf{p}' = e\,\mathbf{E} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}], \tag{6}$$

where $\beta = V/c$. By integrating (2) we obtain

$$\mathbf{H} = \beta^{-1}[\mathbf{i}\times\mathbf{E}] + \mathbf{H}_0, \tag{7}$$

where $\mathbf{H}_0$ is the intensity of the external magnetic field acting on the plasma. If there is no such field, and the plasma performs self-oscillations, then $\mathbf{H} = \beta^{-1}\,\mathbf{i}\times\mathbf{E}$. In that case

$$\mathbf{i}\mathbf{H} = 0, \qquad \mathbf{E}\mathbf{H} = 0. \tag{8}$$

In other words, the magnetic field is in that case transverse and perpendicular to the electric field.

From (3) and (5) we see that

$$n = n_0 V / (V - \mathbf{i}\mathbf{v}). \tag{9}$$

Since $n > 0$ it follows that $\mathbf{i}\cdot\mathbf{v} < V$, i.e., the component of the electron velocity in the direction of propagation of the wave must always be less than the phase velocity.

We multiply (6) on the left vectorially by i and use (7). This gives

$$\mathbf{H} = -\frac{c}{e}[\mathbf{i}\times\mathbf{p}'] + \frac{V\mathbf{H}_0 - \mathbf{v}(\mathbf{i}\mathbf{H}_0)}{V - \mathbf{i}\mathbf{v}}. \tag{10}$$

Next we multiply (3) on the left vectorially by i and find

$$\mathbf{H}' = (4\pi/c)\, en\, (\beta^2 - 1)^{-1}[\mathbf{i}\times\mathbf{v}] \tag{11}$$

From (10) and (11) it follows that

$$[\mathbf{i}\times\mathbf{p}]'' + \frac{4\pi e^2 n}{(\beta^2-1)c^2}[\mathbf{i}\mathbf{v}] = \frac{e}{c}\left(\frac{V\mathbf{H}_0 - \mathbf{v}(\mathbf{i}\mathbf{H}_0)}{V - \mathbf{i}\mathbf{v}}\right)'. \tag{12}$$

Taking the scalar product of (6) with i and using (5) we find

$$\left((\mathbf{i}\mathbf{v} - V)\,\mathbf{i}\mathbf{p}' + [\mathbf{i}\mathbf{v}]\,[\mathbf{i}\mathbf{p}]' - e\beta\frac{\mathbf{i}\,[\mathbf{v}\mathbf{H}_0]}{V - \mathbf{i}\mathbf{v}}\right)' \tag{13}$$

$$\operatorname{div} \mathbf{E} = 4\pi e (n -$$

The Eqs. (12) and (13) determine the transverse and longitudinal velocity components in the general case when $\mathbf{H}_0$ does not vanish.

Taking the vector i in the z direction, and introducing the dimensionless momentum $\rho = \mathbf{P}/mc$ and the dimensionless velocity $\mathbf{u} = \mathbf{v}/c$, we bring (12) and (13) to the form

$$\frac{d^2\rho_x}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta}{\beta-u_z}u_x + \beta\frac{d}{d\tau}\frac{\beta\Omega_y - u_y\Omega_z}{\beta - u_z} = 0;$$

$$\frac{d^2\rho_y}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta}{\beta-u_z}u_y - \beta\frac{d}{d\tau}\frac{\beta\Omega_x - u_x\Omega_z}{\beta - u_z} = 0;$$

$$\frac{d}{d\tau}\left\{(u_z - \beta)\frac{d\rho_z}{d\tau} + u_x\frac{d\rho_x}{d\tau}\right.$$

$$\left. + u_y\frac{d\rho_x}{d\tau} + \frac{\beta^2}{\beta - u_z}(u_x\Omega_y - u_y\Omega_x)\right\} = \omega_0^2\frac{\beta^2 u}{\beta - u_z}, \tag{14}$$

where $\tau = t - (\mathbf{i}\cdot\mathbf{r}/V)$, $\omega_0^2 = 4\pi e^2 n_0/m$, $\Omega = e\mathbf{H}_0/mc$:

In the absence of the external field $\mathbf{H}_0$ the equations of motion (14) take the form

$$\frac{d^2\rho_x}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta}{\beta-u_z}u_x = 0;$$

$$\frac{d^2\rho_y}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta}{\beta-u_z}u_y = 0; \qquad (15)$$

$$\frac{d}{d\tau}\left\{(u_z-\beta)\frac{d\rho_z}{d\tau} + u_x\frac{d\rho_x}{d\tau} + u_y\frac{d\rho_y}{d\tau}\right\} = \omega_0^2\frac{\beta^2 u_z}{\beta-u_z}.$$

or

$$\frac{d^2\rho_x}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta\rho_x}{\beta\sqrt{1+\rho^2}-\rho_z} = 0;$$

$$\frac{d^2\rho_y}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta\rho_y}{\beta\sqrt{1+\rho^2}-\rho_z} = 0; \qquad (15')$$

$$\frac{d^2}{d\tau^2}(\beta\rho_z - \sqrt{1+\rho^2}) + \omega_0^2\frac{\beta^2\rho_z}{\beta\sqrt{1+\rho^2}-\rho_z} = 0.$$

The first two equations (15) for the transverse components evidently admit finite solutions only if $\beta > 1$, i.e., $V > c$. As regards the third equation (15) for the longitudinal velocity component u_z, it has finite solutions for arbitrary β, if $u_x = u_y = 0$. Such purely longitudinal motions for $\beta < 1$ are, however, unstable because of the coupling between longitudinal and transverse motion.

If we introduce, in place of the momentum components, the new variables

$$\xi = \sqrt{\beta^2-1}\,\rho_x,$$

$$\eta = \sqrt{\beta^2-1}\,\rho_y, \quad \zeta = \beta\rho_z - \sqrt{1+\rho^2},$$

we obtain (15′) in the form

$$\frac{d^2\xi}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta\xi}{\sqrt{\beta^2-1+\xi^2+\eta^2+\zeta^2}} = 0; \quad (16)$$

$$\frac{d^2\eta}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta\eta}{\sqrt{\beta^2-1+\xi^2+\eta^2+\zeta^2}} = 0;$$

$$\frac{d^2\zeta}{d\tau^2} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta\zeta}{\sqrt{\beta^2-1+\xi^2+\eta^2+\zeta^2}} + \omega_0^2\frac{\beta^2}{\beta^2-1} = 0.$$

These equations can be derived from the Lagrangian

$$L = \frac{1}{2}\left[\left(\frac{d\xi}{d\tau}\right)^2 + \left(\frac{d\eta}{d\tau}\right)^2 + \left(\frac{d\zeta}{d\tau}\right)^2\right] - \omega_0^2\frac{\beta^2}{\beta^2-1}[\beta\sqrt{\beta^2-1+\xi^2+\eta^2+\zeta^2}+\zeta]. \qquad (17)$$

Hence, the general problem of the wave motion of the plasma in the absence of an external magnetic field is equivalent to that of the nonrelativistic motion of a particle of unit mass in a field with potential energy

$$U = \omega_0^2\frac{\beta^2}{\beta^2-1}(\beta\sqrt{\beta^2-1+\xi^2+\eta^2+\zeta^2}+\zeta). \qquad (17')$$

From the form of the Lagrangian follows immediately the laws of conservation of energy and momentum:

$$\frac{1}{2}\left[\left(\frac{d\xi}{d\tau}\right)^2 + \left(\frac{d\eta}{d\tau}\right)^2 + \left(\frac{d\zeta}{d\tau}\right)^2\right] + \omega_0^2\frac{\beta^2}{\beta^2-1}[\beta\sqrt{\beta^2-1+\xi^2+\eta^2+\zeta^2}+\zeta] = \mathcal{E};$$

$$\xi d\eta/d\tau - \eta d\xi/d\tau = M.$$

If we go back from the variables ξ, η, ζ to the fields $\mathbf{E}$ and $\mathbf{H}$, and the velocity $\mathbf{v}$, these laws take the following form:

$$\frac{\beta}{8\pi}(\mathbf{E}^2+\mathbf{H}^2) + \beta n_0\frac{mc^2}{\sqrt{1-v^2/c^2}} - \frac{1}{4\pi}\mathbf{i}\cdot[\mathbf{E}\times\mathbf{H}] = \text{const}; \qquad (18)$$

$$\mathbf{p}\times\mathbf{H} = \text{const}.$$

We note that these relations are simple consequences of our fundamental equations. Indeed, taking the scalar product of (3) with $\mathbf{H}'$, one sees easily that $\mathbf{v}\cdot\mathbf{H}' = 0$, i.e., $\mathbf{p}\cdot\mathbf{H}' = 0$. From the scalar product of $\mathbf{H}$ with (6) we see also that $\mathbf{p}'\cdot\mathbf{H} = 0$. Adding these two results gives us the second part of (18). The first part can be derived in a similar manner.

2. PLASMA OSCILLATIONS OF SMALL AMPLITUDE

Turning now to a detailed investigation of the general equations of motion of the plasma, we shall first consider the well-known case of oscillations of small amplitude. We introduce in place of **u** a new variable $\mathbf{w} = \mathbf{v}/V$, and assume that $w \ll 1$. Then (15) takes the form

$$\frac{d^2 w_x}{d\tau^2} + \omega_\perp^2 \frac{w_x}{1 - w_z} = 0; \tag{19}$$

$$\frac{d^2 w_y}{d\tau^2} + \omega_\perp^2 \frac{w_y}{1 - w_z} = 0;$$

$$\frac{d}{d\tau}\left[(w_z - 1)\frac{dw_z}{d\tau} + w_x \frac{dw_x}{d\tau} + w_y \frac{dw_y}{d\tau}\right] = \omega_0^2 \frac{w_z}{1 - w_z},$$

where $\omega_\perp^2 = \omega_0^2 \beta^2/(\beta^2 - 1)$.

The Eqs. (19) can be solved by successive approximations. If we neglect w_z compared to unity, and also the terms quadratic in **w**, we find in first approximation uncoupled longitudinal and transverse waves, with the frequencies ω_0 and $\omega_\perp$:

$$w_x^{(0)} = W_x \cos \omega_\perp \tau; \tag{20}$$

$$w_y^{(0)} = W_y \sin \omega_\perp \tau; \quad w_z^{(0)} = W_z \cos(\omega_0 \tau + \alpha).$$

Note that the transverse frequency $\omega_\perp$ corresponds to a dielectric constant of the plasma:

$$\varepsilon = 1 - (\omega_0/\omega_\perp)^2.$$

Indeed, if we define the phase velocity as $c/\sqrt{\epsilon}$, and insert the above value of ϵ, we obtain V.

In the next approximation the longitudinal and transverse oscillations are already coupled. To find the velocity components in this approximation, let $\mathbf{w} = \mathbf{w}^{(0)} + \mathbf{w}^{(1)}$ in (19), where $\mathbf{w}^{(0)}$ is given by (20). Retaining terms of order $w^{(0)2}$ and neglecting $w^{(0)} w^{(1)}$ and $w^{(1)2}$, we obtain

$$w_x = w_x^{(0)} + w_x^{(1)} = W_x \cos \omega_\perp \tau - \frac{1}{2} W_x W_z \omega_\perp^2 \frac{\cos[(\omega_0 + \omega_\perp)\tau + \alpha]}{\omega_\perp^2 - (\omega_0 + \omega_\perp)^2} - \frac{1}{2} W_x W_z \omega_\perp^2 \frac{\cos[(\omega_0 - \omega_\perp)\tau + \alpha]}{\omega_\perp^2 - (\omega_0 - \omega_\perp)^2};$$

$$w_y = w_y^{(0)} + w_y^{(1)} = W_y \sin \omega_\perp \tau - \frac{1}{2} W_y W_z \omega_\perp^2 \frac{\sin[(\omega_0 + \omega_\perp)\tau + \alpha]}{\omega_\perp^2 - (\omega_0 + \omega_\perp)^2} - \frac{1}{2} W_y W_z \omega_\perp^2 \frac{\sin[(\omega_0 - \omega_\perp)\tau - \alpha]}{\omega_\perp^2 - (\omega_0 - \omega_\perp)^2}; \tag{20'}$$

$$w_z = w_z^{(0)} + w_z^{(1)} = W_z \cos(\omega_0 \tau + \alpha) - \frac{1}{2} W_z^2 + \frac{\beta^2}{3\beta^3 + 1}(W_x^2 - W_y^2)\cos 2\omega_\perp \tau + \frac{1}{2} W_z^2 \cos 2(\omega_0 \tau + \alpha).$$

We now have obtained anharmonic oscillations, in which the transverse anharmonicity contains the combination frequencies $\omega_\perp + \omega_0$ and $\omega_\perp - \omega_0$, whereas the longitudianl anharmonicity contains the double frequencies $2\omega_0$ and $2\omega_\perp$. Note that the average of w_z does not vanish; however, the average of the current density $n\mathbf{w}$ vanishes.

If the field $\mathbf{H}_0$ does not vanish, oscillations of small amplitude obey [according to (14)] the equations

$$d^2 u_x/d\tau^2 + \Omega_y du_z/d\tau - \Omega_z du_y/d\tau + \beta^2(\beta^2 - 1)^{-1}\omega_0^2 u_x = 0;$$

$$d^2 u_y/d\tau^2 + \Omega_z du_x/d\tau - \Omega_x du_z/d\tau + \beta^2(\beta^2 - 1)^{-1}\omega_0^2 u_y = 0;$$

$$d^2 u_z/d\tau^2 + \Omega_x du_y/d\tau - \Omega_y du_x/d\tau + \omega_0^2 u_z = 0.$$

Setting $u_x = U_x e^{i\omega t}$, etc., we find the well-known expression for the refractive index

$$\frac{1}{\beta^2_{1,2}} = 1 - \omega_0^2\left[\omega^2 + \frac{\Omega^2\omega^2\sin^2\vartheta}{2(\omega_0^2-\omega^2)} \pm\left(\frac{\Omega^4\omega^4\sin^4\vartheta}{4(\omega_0^2-\omega^2)^2} + \Omega^2\omega^2\cos^2\vartheta\right)^{1/2}\right]^{-1}, \quad (21)$$

where ϑ is the angle between the field and the direction of propagation.

Since β^2 must be real, we can determine from this the range of frequencies which may be propagated in the plasma. We see from (21) that in the presence of an external magnetic field the quantity β may be greater, as well as less than, unity.

3. LONGITUDINAL OSCILLATIONS

Next we consider longitudinal plasma oscillations, without assuming the amplitude to be small. Assuming in (16) that $u_x = u_y = 0$, we find

$$(d/d\tau)\{(u-\beta)\,d\rho/d\tau\} = \omega_0^2\beta^2 u/(\beta - u),$$

where $u = u_z$ and $\rho = \rho_z$ are the dimensionless velocity and momentum of the electron. By expressing the momentum in terms of the velocity we can bring this equation into the form

$$\frac{d^2}{d\tau^2}\frac{1-\beta u}{\sqrt{1-u^2}} = \omega_0^2\frac{\beta^2 u}{\beta - u}.$$

If we now multiply by $(d/d\tau)[(1-\beta u)/(\sqrt{1-u^2})]$ and integrate, we find

$$\frac{1}{2}\left(\frac{d}{d\tau}\frac{1-\beta u}{\sqrt{1-u^2}}\right)^2 = \beta^2\omega_0^2\left(C - (1-u^2)^{-1/2}\right),$$

where C is a constant of integration. Putting $C = (1-u_m^2)^{-1/2}$, we see that u must lie in the interval $-u_m \leq u \leq u_m$. By a further integration we find

$$\int_0^u [(1-u_m^2)^{-1/2} - (1-u^2)^{-1/2}]^{-1/2}\frac{(\beta-u)\,du}{(1-u^2)^{3/2}} = \sqrt{2}\,\beta\omega_0\tau. \quad (22)$$

This formula solves our problem in principle, since it expresses u as a function of $\tau = t - (z/V)$.

By introducing in place of u a new independent variable λ,

$$u = \tanh\lambda,\ u_m = \tanh\lambda_0,$$

one can put (22) in the form

$$\int_0^\lambda \frac{\beta\cosh\lambda - \sinh\lambda}{\sqrt{\cosh\lambda_0 - \cosh\lambda}}\,d\lambda = \sqrt{2}\,\beta\omega_0\tau. \quad (22')$$

Evidently u is a periodic function of τ. Its period, which we shall denote by T is determined by the equation

$$2\int_{-u_m}^{+u_m}[(1-u_m^2)^{-1/2} - (1-u^2)^{-1/2}]^{-1/2}\frac{(\beta-u)\,du}{(1-u^2)^{3/2}} = \sqrt{2}\,\beta\omega_0 T.$$

Or, in terms of the frequency $\omega = 2\pi/T$:

$$\omega = \omega_0\frac{\pi}{\sqrt{2}\,I(u_m)},\quad I(u_m) = \int_0^{u_m}\left[\left(\frac{1-u^2}{1-u_m^2}\right)^{1/2} - 1\right]^{-1/2}\frac{du}{(1-u^2)^{5/4}}. \quad (23)$$

We thus see that the period of longitudinal oscillations depends on the velocity amplitude u_m.

Simple expressions can be obtained in the two limiting cases of small and large velocity amplitude. In the first case, $u_m \ll 1$, the frequency is

$$\omega = \omega_0\left(1 - \frac{3}{16}u_m^2\right),\quad u_m \ll 1. \quad (24)$$

In the second case, $1 - u_m \ll 1$, the frequency becomes

$$\omega = 2^{-3/2}\pi\omega_0(1-u_m^2)^{1/4},\quad 1 - u_m \ll 1. \quad (25)$$

As $u_m \to 0$, the frequency tends to zero. This is connected with the fact that for $u_m \to 1$ the electron mass tends to infinity.

In the general case of intermediate values of u_m the integral in (22) and the period can be expressed in terms of elliptic functions. Indeed, with the substitution

$$u = \frac{2k\,\mathrm{sn}(x;k)}{1 + k^2\,\mathrm{sn}^2(x;k)},$$

$$k = \left[\frac{1-\sqrt{1-u_m^2}}{1+\sqrt{1-u_m^2}}\right]^{1/2},\quad k' = \sqrt{1-k^2},$$

we find

$$\frac{k}{\beta k'}\frac{\operatorname{cn}(x;k)}{\operatorname{dn}(x;k)} - k'x + \frac{1}{k'K}\left[2E_x + \frac{\vartheta_3'(x/2K;q)}{\vartheta_3(x/2K;q)}\right] = \omega_0\tau, \quad (26)$$

where

$$\frac{\vartheta_3'(z;q)}{\vartheta_3(z;q)} = 4\pi\sum_{n=1}^{\infty}\frac{(-1)^n q^n \sin 2\pi nz}{1-q^{2n}}; \quad q = e^{-\pi k'/K};$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1+x^2)(1-k^2x^2)}};$$

$$K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}};$$

$$E = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx.$$

Setting in (26) $u = u_m$, $\tau = 1/4\ T$, and noting that $x = K$ when $u = u_m$, we find the following expression for the oscillation frequency

$$\omega = \omega(\pi/2)\,k'/(2E - k'^2K). \quad (27)$$

From u we can easily obtain the particle density, and the electric field, which is in the direction of propagation. From (6) and (9), we find

$$n(\tau) = \frac{n_0\beta}{\beta - u}; \quad eE(\tau) = \pm\sqrt{2}\,m\omega_0 c\left[\frac{1}{\sqrt{1-u_m^2}} - \frac{1}{\sqrt{1-u^2}}\right]^{1/2}. \quad (28)$$

The maximum value of the field is, for small amplitude, proportional to u_m; for $1 - u_m \ll 1$, it is given by

$$eE_m = \sqrt{2}\,m\omega_0 c\,(1-u_m^2)^{1/4}, \quad (28')$$

If $1 - u_m \ll 1$,

$$u = \begin{cases} u_m & \text{for} \quad 0 < \theta < 2\sqrt{2}(1 - 1/2\beta); \\ -u_m & \text{for} \quad -2\sqrt{2}(1 + 1/2\beta) < \theta < 0, \end{cases}$$

where $\theta = \omega_0\,\tau(1-u_m^2)^{1/4}$. Using these expressions we may consider almost longitudinal oscillations in the case $1 - u_m \ll 1$. They are described according to (14) by the equation

$$d^2\rho_x/d\theta^2 + f(\theta)\,\rho_x = 0$$

with a given function $f(\theta)$. This function $f(\theta)$ may be approximately expressed in the form

$$f(\theta) = f_0(\theta) + h\Big\{\sum_n \delta(\theta - 4\sqrt{2}n) + \sum_n \delta\Big(\theta - 2\sqrt{2}\Big(1 - \frac{1}{2\beta}\Big) - 4\sqrt{2}n\Big)\Big\}.$$

Here $f_0(\theta)$ is a periodic function, defined by the relations

$$f_0(\theta) = \frac{\beta^2}{\beta^2-1}\,\frac{\beta}{\beta-1} \quad \text{for} \quad 0 < \theta < 2\sqrt{2}\Big(1 - \frac{1}{2\beta}\Big);$$

and

$$f_0(\theta) = \frac{\beta^2}{\beta^2-1}\,\frac{\beta}{\beta+1} \quad \text{for} \quad -2\sqrt{2}\Big(1 + \frac{1}{2\beta}\Big) < \theta < 0;$$

$$h = \frac{\sqrt{2}\beta^2}{\beta^2-1}\Big(2\ln\frac{2\sqrt{2}}{(1-u_m^2)^{1/4}} - \frac{2\beta^2-1}{\beta^2-1}\Big).$$

The solution of this equation is of the form $\rho_x = e^{ik\theta}\varphi(\theta)$ where $\varphi(\theta)$ is a periodic function and k is defined by

$$\cos 4\sqrt{2}\,k = \cos k_1a_1 \cos k_2a_2 - \frac{k_1^2 + k_2^2 - h^2}{2k_1k_2}\sin k_1a \sin k_2a - \frac{h}{k_1}\sin k_1a_1 \cos k_2a_2 - \frac{h}{k_2}\sin k_2a_2 \cos k_1a_1;$$

$$a_1 = \sqrt{2}\,\frac{2\beta-1}{\beta}; \quad a_2 = \sqrt{2}\,\frac{2\beta+1}{\beta};$$

$$k_1^2 = \frac{\beta^2}{\beta^2-1}\,\frac{\beta}{\beta-1}; \quad k_2^2 = \frac{\beta^2}{\beta^2-1}\,\frac{\beta}{\beta+1}.$$

4. TRANSVERSE OSCILLATIONS

For purely transverse oscillations $u_z = 0$, and the third Eq. (15′) gives $p^2 = \text{const}$. This follows also with H_0 present, provided H_0 is parallel to the direction of propagation.

Putting $\rho_z = 0$ in the first two Eqs. (15′) we obtain

$$\rho_x = \rho \cos \omega\tau; \quad \rho_y = \rho \sin \omega\tau,$$

where

$$\omega = \omega_0 \beta (\beta^2 - 1)^{-1/2} (1 + \rho^2)^{-1/4}. \qquad (29)$$

Therefore, the wave velocity β may be expressed in the form $\beta = \epsilon^{-1/2}$, where

$$\varepsilon = 1 - \frac{\omega_0'^2}{\omega^2};$$

$$\omega_0' = \sqrt{\frac{4\pi e^2 n_0}{m'}}; \quad m' = \frac{m}{\sqrt{1 - v^2/c^2}}$$

(v is the electron velocity).

Thus in the case of purely transverse waves the electrons move around in circular orbits with angular velocity ω, and there can exist only waves with circular polarization*. This is connected with the fact that for large amplitude one cannot superimpose two oscillations of opposite circular polarization because of the nonlinearity of the equations. For small amplitude, when the oscillations are linear, such a superposition is possible, and this leads to the appearance of linearly polarized oscillations, as has been shown.

From (10) (with $H_0 = 0$) and (29) it is easily shown that the magnetic field H is parallel to the electron momentum, and is given by the relations

$$eH_x = -(mc\omega/\beta)\rho_x = -(mc\omega\rho/\beta)\cos\omega\tau;$$

$$eH_y = -(mc\omega/\beta)\rho_y = -(mc\omega\rho/\beta)\sin\omega\tau.$$

The electric field is, according to (7),

$$eE_x = mc\omega\rho \sin\omega\tau; \quad eE_y = -mc\omega\rho\cos\omega\tau.$$

If the external magnetic field is not zero, and is parallel to the direction of propagation, the equations of motion for transverse oscillations take the form

$$\frac{d^2\rho_x}{d\tau^2} - \Omega\frac{du_y}{d\tau} + \frac{\beta^2}{\beta^2-1}\omega_0^2 u_x = 0;$$

$$\frac{d^2\rho_y}{d\tau^2} + \Omega\frac{du_x}{d\tau} + \frac{\beta^2}{\beta^2-1}\omega_0^2 u_y = 0,$$

where, as before, $u_x^2 + u_y^2 = \text{const}$. Remembering that $\rho = u(1 - u^2)^{-1/2}$ and putting $u_x = U\cos\omega\tau$, $u_y = U\sin\omega\tau$, we find the following expression for the frequency ω:

$$\omega = \left[\frac{\Omega'^2}{4} + \frac{\beta^2}{\beta^2-1}\omega_0'^2\right]^{1/2}, \qquad (30)$$

and, therefore,

$$\beta^{-2} = 1 - \omega_0'^2/(\omega^2 \pm \omega\Omega),$$

where

$$\omega_0'^2 = \omega_0^2\sqrt{1-u^2}, \quad \Omega' = \Omega\sqrt{1-u^2}.$$

Note that the present expression for β^{-2} is formally identical with (21) for $\vartheta = 0$. However, the difference consists in the fact that (21) refers to linear oscillations, and one may therefore superimpose two waves for which the sign of the root in (21) is different, whereas the nonlinear oscillations to which (30) relates, may not be so superimposed. In this case we may therefore either have the solution with the positive or that with the negative sign in the expression for the frequency.

We now turn to the case of almost transverse oscillations, for which the electron trajectories are almost circles. For this purpose we replace ρ_x and ρ_y in the fundamental Eq. (18) by the new variables $\rho_\perp$ and φ, related to ρ_x and ρ_y by the relation $\rho_x + i\rho_y = \rho_\perp e^{i\varphi}$. The first two Eqs. (15′) can then be written in the form

$$\frac{d^2\rho_\perp}{d\tau_2} - \rho_\perp\left(\frac{d\varphi}{d\tau}\right)^2 + \frac{\beta^2}{\beta^2-1}\omega_0^2\frac{\beta\rho_\perp}{\beta\sqrt{1+\rho_\perp^2+\rho_z^2} - \rho_z} = 0;$$

$$2\rho_\perp\frac{d\rho_\perp}{d\tau}\frac{d\varphi}{d\tau} + \rho_\perp^2\frac{d^2\varphi}{d\tau^2} = 0.$$

Integration of the second of these gives

$$\rho_\perp^2\, d\varphi/d\tau = M,$$

where M is a constant. Therefore, the first equation becomes

$$\frac{d^2\rho_\perp}{d\tau^2} - \frac{M^2}{\rho_\perp^3} + \omega_0^2\frac{\beta^2}{\beta^2-1}\frac{\beta\rho_\perp}{\beta\sqrt{1+\rho_\perp^2+\rho_z^2}-\rho_z} = 0. \qquad (31)$$

* This argument loses its usefulness in practice when $\beta \gg 1$ since, because of the small longitudinal velocity component, almost transverse oscillations then become possible which are linearly polarized (see Sec. 5 below).

If we put here $\rho_z = 0$, we obtain transverse oscillations with constant $\rho_\perp$. If we call this constant value ρ_0, we find from (31)

$$\omega_\perp^2 = \omega_0^2\beta^2/(\beta^2-1). \tag{32}$$

Consider now small oscillations of the quantity $\rho_\perp$ about the value ρ_0. Letting $\rho_\perp = \rho_0 + \delta$ and assuming δ and ρ_z small compared to ρ_0, we obtain from (31) with the help of (32):

$$\frac{d^2\delta}{d\tau^2} + \frac{\omega_\perp^2(4+3\rho_\perp^2)}{(1+\rho_0^2)^{3/2}}\delta + \frac{\omega_\perp^2\rho_0\rho_z}{\beta(1+\rho_0^2)} = 0.$$

By a similar expansion in the third Eq. (15′) we obtain a second equation for δ and ρ_0:

$$\beta\frac{d^2\rho_z}{d\tau^2} - \frac{\rho_0}{\sqrt{1+\rho_0^2}}\frac{d^2\delta}{d\tau^2} + \omega_0^2\frac{\beta\rho_z}{\sqrt{1+\rho_0^2}} = 0.$$

Putting $\delta = De^{i\omega t}$, $\rho_z = Re^{i\omega t}$, we find the following equations for the frequencies of coupled transverse-longitudinal oscillations:

$$\omega^4 - \frac{\omega_0^2(4\beta^2\rho_0^2+5\beta^2-1)}{(\beta^2-1)(1+\rho_0^2)^{3/2}}\omega^2 - \frac{\omega_0^4\beta^2(4+3\rho_0^2)}{(\beta^2-1)(1+\rho_0^2)^2} = 0,$$

and hence

$$\omega_{1,2} = \frac{\omega_\perp}{\sqrt{1+\rho_0^2}}\left\{\frac{4\beta^2\rho_0^2+5\beta^2-1}{2\beta\sqrt{1+\rho_0^2}} \pm \left[\left(\frac{4\beta^2\rho_0^2+5\beta^2-1}{2\beta^2\sqrt{1+\rho_0^2}}\right)^2 - \frac{(\beta^2-1)(4+3\rho_0^2)}{\beta^2}\right]^{1/2}\right\}^{1/2} \tag{33}$$

Some limiting cases are of interest.

If $\rho \ll 1$,

$$\omega_1 = 2\omega_\perp, \qquad \omega_2 = \omega_0, \tag{34}$$

in agreement with the result found previously for the frequencies of longitudinal and transverse oscillations in the linear approximation. (We have found here twice the value of $\omega_\perp$, since we are here concerned with oscillations of the resultant, which evidently have a period equal to one-half that of the oscillations of ρ_x and ρ_y.)

If $\rho_0 \gg 1$,

$$\omega_{1,2} = \omega_\perp\rho_0^{-1/2}[2 \pm \beta^{-1}(3+\beta^2)^{1/2}]^{1/2}. \tag{35}$$

In the case $\beta - 1 \ll 1$, the frequencies of coupled oscillations are, for arbitrary ρ_0, given by

$$\omega_1 = 0, \quad \omega_2 = 2\omega_\perp(1+\rho_0^2)^{-1/4}. \tag{36}$$

If $\beta \gg 1$, then for any ρ_0

$$\omega_1 = \omega_0\frac{(4+3\rho_0^2)^{1/2}}{(1+\rho_0^2)^{3/4}}; \quad \omega_2 = \frac{\omega_0}{(1+\rho_0^2)^{1/4}}. \tag{37}$$

5. COUPLED LONGITUDINAL-TRANSVERSE OSCILLATIONS

In the preceding sections we considered longitudinal, transverse and almost transverse plasma oscillations. The investigation of the general case which might be called the case of longitudinal-transverse oscillations, amounts to the integration of Eq. (16) and represents a very complicated problem, which is soluble only in some limiting cases, viz., for very large β and for β close to unity*.

Consider first the case $\beta \gg 1$. We shall assume that β^2 and $\xi^2+\eta^2+\zeta^2$ are quantities of the same order. (If $\xi^2+\eta^2+\zeta^2 \ll \beta^2$, then we get back to the case of small oscillations, which has already been discussed, since in that case ξ, η, ζ are proportional to ρ_x, ρ_y, ρ_z**.) With these assumptions we may neglect the term ζ in the potential energy U defined by (17′), which determines the motion of the plasma. The problem therefore reduces to the integration of the equations of motion for a particle in a central field whose Lagrangian is, according to (17),

$$L = \frac{1}{2}\left[\left(\frac{d\xi}{d\tau}\right)^2 + \left(\frac{d\eta}{d\tau}\right)^2 + \left(\frac{d\zeta}{d\tau}\right)^2\right] - \omega_0^2\beta\sqrt{\beta^2+\xi^2+\eta^2+\zeta^2}.$$

We replace $\xi, \eta, \zeta, \tau, L$ by the new variables $X, Y, Z, \theta, \mathcal{L}$, defined by $X = \xi/\beta$, $Y = \eta/\beta$, $Z = \zeta/\beta$, $\theta = \omega_0\tau$, $\mathcal{L} = \omega_0^2\beta^2 L$ and find

$$\mathcal{L} = \frac{1}{2}\left\{\left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2 + \left(\frac{dZ}{d\theta}\right)^2\right\} - \sqrt{1+R^2},$$

* The possibility of a solution for these cases was pointed out to us by L. D. Landau.

** From $\rho_z = [\beta\zeta + (\beta^2 - 1 + \xi^2+\eta^2+\zeta^2)^{1/2}](\beta^2-1)^{-1}$ follows, if $\beta^2 \gg \xi^2+\eta^2+\zeta^2$ and $\beta^2 \gg 1$, that $\rho_z = \zeta + 1/\beta$. In addition, $\rho_x = \xi/\beta$ and $\rho_y = \eta/\beta$.

where $R^2 = X^2 + Y^2 + Z^2$. Since the motion in this problem takes place in a plane, it is convenient to rotate the coordinate axes so as to make the X-Y plane perpendicular to the direction of the angular momentum. The Lagrangian then takes the form:

$$\mathscr{L} = \frac{1}{2}\{(dR/d\theta)^2 + R^2(d\varphi/d\theta)^2\} - \sqrt{1+R^2},$$

where φ is the polar angle. We write down the energy and angular momentum equations

$$\frac{1}{2}\left(\frac{dR}{d\theta}\right)^2 + \frac{M^2}{2R^2} + \sqrt{1+R^2} = \mathscr{E}; \qquad R^2\frac{d\varphi}{d\theta} = M,$$

and hence find finally the solution

$$\omega_0\tau = \int\left[2\mathscr{E} - \frac{M^2}{R^2} - 2\sqrt{1+R^2}\right]^{-1/2} dR. \quad (38)$$

The quantity R which occurs here is connected with the dimensionless momentum ρ by

$$R^2 = \rho^2 - (2\rho_z/\beta)\sqrt{1+\rho^2} + \beta^{-2}.$$

For large ρ, R practically coincides with ρ. We see from (38) that the quantity R oscillates between two limits R_0 and R_1, which are related to $\mathscr{E}$ and M by

$$M^2 = \frac{2R_0^2R_1^2}{\sqrt{1+R_1^2} + \sqrt{1+R_0^2}};$$

$$\mathscr{E} = \frac{R_0^2 + R_1^2 + 1 + \sqrt{(1+R_0^2)(1+R_1^2)}}{\sqrt{1+R_0^2} + \sqrt{1+R_1^2}}.$$

The frequency of oscillation is

$$\omega = \frac{\pi\omega_0}{\sqrt{2}\, I(R_0, R_1)}, \qquad I(R_0, R_1) \quad (39)$$

$$= \int_{R_0}^{R_1}\left[\mathscr{E} - \frac{M^2}{2R^2} - \sqrt{1+R^2}\right]^{-1/2} dR.$$

For $R_0 = 0$, this integral becomes identical with the integral $I(u_m)$ which gave the frequency of longitudinal vibrations, if we put $R_1 = u_m(1-u_m^2)^{-1/2}$. However, the case considered here does not, for $R_0 = 0$, reduce to the case of purely longitudinal oscillations which we had considered before, since we may here be dealing with oscillations which are almost linearly polarized in an arbitrary direction. In particular, we may have oscillations which are approximately transverse and linearly polarized, and for which ρ_x is non-zero, $\rho_y = 0$, and ρ_z is non-zero, but much less than ρ_x, of the order $\rho_z \sim \rho_x/\beta$. The existence of such oscillations does not conflict with the statement made earlier that purely transverse oscillations have circular rather than linear polarization. Indeed, it is clear from the third part of (15) that for $\beta \gg 1$, $\rho_x \sim 1$ and $\rho_z \sim 1/\beta$ the quantity ρ_y may vanish, i.e., the oscillation may be approximately transverse and linearly polarized.

If $R_0 = R_1$ (R = const) the vector $\mathbf{R}$ describes a circle with angular velocity

$$\omega = \omega_0(1+R^2)^{-1/4}.$$

This formula agrees with (29) for the frequency of transverse oscillations for $\beta \gg 1$. The present type of oscillation is then for $R_0 = R$, very similar to oscillations with circular polarization, for which, however, the plane of oscillation is not necessarily at right angles to the direction of propagation.

We now turn to the case when β is close to unity, $\beta - 1 \ll 1$. The basic Eqs. (15′) may then be written in the form

$$\frac{d^2\rho_x}{d\theta^2} + \frac{\rho_x}{\sqrt{1+\rho^2} - \rho_z} = 0; \quad (40)$$

$$\frac{d^2\rho_y}{d\theta^2} + \frac{\rho_y}{\sqrt{1+\rho^2} - \rho_z} = 0;$$

$$\frac{d^2}{d\theta^2}\left(\rho_z - \sqrt{1+\rho^2}\right) + \frac{(\beta^2-1)\rho_z}{\sqrt{1+\rho^2} - \rho_z} = 0,$$

where $\theta = \omega_0\tau(\beta^2 - 1)^{-1/2}$. Neglecting the last term in the third equation, we find

$$\sqrt{1+\rho^2} - \rho_z = C^2, \quad (40')$$

where C is a constant. The first two Eqs. (40) then take the form

$$\frac{d^2\rho_x}{d\theta^2} + \frac{\rho_x}{C^2} = 0; \qquad \frac{d^2\rho_y}{d\theta^2} + \frac{\rho_y}{C^2} = 0,$$

and hence

$$\rho_x = R_x\cos(\theta/C); \quad (41)$$

$$\rho_y = R_y\sin(\theta/C).$$

Inserting these expressions in (40′) we find

$$\rho_z = \frac{1}{4C^2}\left\{R_x^2 + R_y^2 - 2(C^4 - 1) - (R_x^2 - R_y^2)\cos\frac{2\theta}{C}\right\}. \quad (41')$$

The constants C, R_x and R_y are connected by a relation which follows from the fact that nv_z vanishes on the average. [This condition follows from the fact that $\mathbf{E}'$ and $\mathbf{H}'$ vanish on the average, see (3).] Noting that $u_z = (1+\rho^2)^{-1/2}$ and using (9) and (40′), we find for $\beta - 1 \ll 1$:

$$nu_z \approx \frac{n_0 u_z}{1-u_z} = \frac{n_0\rho_z}{\sqrt{1+\rho^2} - \rho_z} = \frac{n_0}{C^2}\rho_z.$$

Since the average of nu_z vanishes, it follows therefore that also the average of ρ_z must vanish. We can therefore put in (41′)

$$R_x^2 + R_y^2 = 2(C^4 - 1),$$

$$C^2 = \sqrt{1 + {}^1/_2(R_x^2 + R_y^2)}.$$

Finally, ρ_x, ρ_y, ρ_z take the form

$$\rho_x = R_x\cos\omega\tau; \qquad \rho_y = R_y\sin\omega\tau; \quad (42)$$

$$\rho_z = \frac{R_x^2 - R_y^2}{4\sqrt{1 + {}^1/_2(R_x^2 + R_y^2)}}\cos 2\omega\tau,$$

where

$$\omega = \omega_0(\beta^2 - 1)^{-1/2}\left[1 + {}^1/_2(R_x^2 + R_y^2)\right]^{-1/4}. \quad (42')$$

These results agree with the Eqs. (29) and (36) which describe oscillations which are approximately perpendicular with circular polarization. Indeed, when $R_x \approx R_y$, the frequency of oscillation of the quantity R_z takes the value given by (36).

We may further consider the case of high energies, when the inequality $\xi^2 + \eta^2 + \zeta^2 \gg \beta^2 - 1$ is satisfied, β being arbitrary ($\beta > 1$). The Lagrangian of the plasma motion can then be written in the form

$$L = \frac{1}{2}\left[\left(\frac{d\xi}{d\tau}\right)^2 + \left(\frac{d\eta}{d\tau}\right)^2 + \left(\frac{d\zeta}{d\tau}\right)^2\right] - \omega_0^2\frac{\beta^2}{\beta^2-1}\left[\beta\sqrt{\xi^2+\eta^2+\zeta^2} + \zeta\right].$$

Under the substitution

$$\xi = \mu\xi', \quad \eta = \mu\eta', \quad \zeta = \mu\zeta', \quad \tau = \sqrt{\mu}\,\tau',$$

where μ is an arbitrary constant, the Lagrangian is multiplied by μ. This shows that, if the motion $\xi = \xi(\tau)$; $\eta = \eta(\tau)$; $\zeta = \zeta(\tau)$ is possible, then also the similar motion $\xi' = \xi(\tau')$; $\eta' = \eta(\tau')$; $\zeta' = \zeta(\tau')$ is possible. In particular, we can deduce from this a definite dependence of the oscillation frequency on the quantity p_0 which characterizes the electron momentum. The result is that the frequency must be inversely proportional to $\sqrt{p_0}$:

$$\omega = \mathrm{const}\, p_0^{-1/2}. \quad (43)$$

This formula is in agreement with the results obtained previously for the frequency in the region of high energies [Eqs. (25), (29), (35) and (42′)].

In conclusion, the authors would like to express their gratitude to L. D. Landau for valuable advice and for his interest in the work, and to N. I. Akhiezer and I. B. Fainberg for valuable discussions.

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Translated by R. E. Peierls
185