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Shape of the Spectral Line of a Generator with Fluctuating Frequency

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The present paper is a continuation and further development of Bershtein's work[1-3] on the natural line width, and Gorelik's work[4,5] on the "technical" width of a vacuum tube generator.

IN any actual vacuum tube generator there are always fluctuations of the frequency (or phase) which produce a smearing of the spectral line. If we consider fluctuations arising from shot or thermal noise (i.e., from "natural" causes), then we speak of the natural line width. If on the other hand we consider "technical" causes of fluctuation, such as "microscopic" effects, then we speak of the technical line width. It is of interest to consider the general question of the relation of the shape of the spectral line of a generator to various characteristics of the frequency fluctuations, which depend on the method of generation of the fluctuations. The present paper treats this problem.

1. THE GENERAL EXPRESSION FOR THE SPECTRAL DENSITY OF THE OSCILLATION

We consider an oscillation whose frequency fluctuates:

$$z_t = A\cos(\omega_0 t + \varphi_t), \tag{1}$$

$$\varphi_t = \int_{t_0}^{t} \Delta\omega_\xi \, d\xi. \tag{2}$$

A is the amplitude, ω_0 is the average value of the frequency, φ_t is the phase of the oscillation and $\Delta\omega_t$ is the frequency fluctuation, whose average value is zero.

The spectral density $S_z(\omega)$ is equal to

$$S_z(\omega) = \frac{2}{\pi}\int_0^\infty \Phi_z(\tau)\cos\omega\tau \, d\tau, \tag{3}$$

where

$$\Phi_z(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T z_t z_{t+\tau}\, dt \tag{4}$$

is the correlation function for the quantity z_t. Substituting (1) in (4), we obtain, after some simple transformations:

$$\Phi_z(\tau) = \lim_{T\to\infty}\frac{A^2}{2T}\int_0^T \cos(\omega_0\tau + \Delta\varphi)\, dt,$$

where

$$\Delta\varphi = \varphi_{t+\tau} - \varphi_t = \int_t^{t+\tau}\Delta\omega_\xi\, d\xi. \tag{5}$$

We shall assume that the frequency fluctuations represent a stationary random process and are a superposition of a large number of random, statistically independent, disturbances. In this case, $\Delta\varphi$ also describes a stationary process and, in addition, has a normal distribution around zero value. Consequently, on the basis of the quasi-ergodic theorem[6], the time average of a function of $\Delta\varphi$ can be identified with the statistical average over the corresponding ensemble (which we shall denote by a bar over the symbol which is averaged). Thus,

$$\Phi_z(\tau) = {}^1/_2 A^2 \overline{\cos(\omega_0\tau + \Delta\varphi)} \tag{6}$$
$$= {}^1/_2 A^2 \cos\omega_0\tau\, \overline{\cos\Delta\varphi},$$

since

$$\overline{\sin\Delta\varphi} = 0.$$

To find the average value of the cosine we use the well-known formula

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\cos qx\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-q^2/4\alpha}$$

and obtain

$$\overline{\cos\Delta\varphi} = \exp(-{}^1/_2\overline{\Delta\varphi^2}). \tag{7}$$

Combining (7) and (6), we find

$$\Phi_z(\tau) = {}^1/_2 A^2 \cos\omega_0\tau \exp(-{}^1/_2\overline{\Delta\varphi^2}). \tag{8}$$

To get $\overline{\Delta\varphi^2}$ we square both sides of (5) and average them:

$$\overline{\Delta\varphi^2} = \int_t^{t+\tau}\int_t^{t+\tau} \overline{\Delta\omega_\xi\,\Delta\omega_\eta}\, d\xi\, d\eta$$
$$= \int_t^{t+\tau}\int_t^{t+\tau} \Phi(\xi-\eta)\, d\xi\, d\eta,$$

where $\Phi(\tau)$ is the correlation function of the frequency fluctuations, which we shall assume to be given. If we change variables in the double integral and use the fact that the correlation function is even, we get

$$\overline{\Delta\varphi^2} = 2\int_0^\tau (\tau-\xi)\,\Phi(\xi)\, d\xi. \tag{9}$$

We write the correlation function in the form

$$\Phi(\tau) = \overline{\Delta\omega^2}\, R(\tau), \tag{10}$$

where $R(\tau)$ is the correlation coefficient of the frequency fluctuation $\Delta\omega_t$ and $\overline{\Delta\omega^2}$ is the mean square deviation of the frequency.

Combining (10), (9) and (7), we get the final expression for the correlation function:

$$\Phi_z(\tau) \tag{11}$$
$$= {}^1/_2 A^2 \cos\omega_0\tau \exp\left[-\overline{\Delta\omega^2}\int_0^\tau (\tau-\xi)\,R(\xi)\, d\xi\right].$$

We shall discuss some of the features of this result. Formula (9) shows that $\overline{\Delta\varphi^2}$ does not depend on the time coordinate t, but depends only on the time interval τ, i.e., the distribution of $\Delta\varphi$ actually does not depend on the time, but is stationary. However, this is not the case for the phase φ_t itself or the quantity $\overline{\varphi_t^2}$, which do depend on the time. This means that the distribution of the phase, though normal, is not stationary.

The problem of finding the spectrum of the oscillations in the presence of frequency fluctuations has also been considered by Middleton[7]. For example, he treated the case where the spectral density of the frequency fluctuations has a Gaussian shape. However, in our opinion, Middleton made the mistake in his calculations of setting $\overline{\varphi_t^2} \equiv \overline{\varphi_{t+\tau}^2}$. Actually, one can show that even for small τ,

$$\overline{\varphi^2{}_{t+\tau}} - \overline{\varphi^2{}_t} = 2\tau\int_0^\infty \Phi(\xi)\, d\xi \not\equiv 0.$$

By finding the parameters of the normal probability distribution for the quantity φ_t, one can show that the phase fluctuations follow a diffusion law, which clearly should be the case, since the phase shifts due to frequency fluctuations pile up [because of the time integral in Eq. (2)]. This diffusion law for the phase fluctuations was considered in more detail earlier by Bershtein in the papers mentioned, in which he investigated the phase fluctuations of a tube generator arising from shot and thermal noise*.

Substituting the value (11) for the correlation function in Eq. (3), we find the spectral density

$$S_z(\omega) = S(\omega) + S(-\omega),$$

where

$$S(\omega) = \frac{A^2}{2\pi}\int_0^\infty \cos(\omega_0-\omega)\tau \tag{12}$$
$$\times \exp\left[-\overline{\Delta\omega^2}\int_0^\tau (\tau-\xi)\,R(\xi)\, d\xi\right] d\tau.$$

The spectral density in the neighborhood of the average frequency ω_0 is determined mainly by the first term $S(\omega)$, since it had a singularity just at the point $\omega = \omega_0$. The second integral $S(-\omega)$ produces no essential change in the spectral density $S_z(\omega)$ at $\omega = \omega_0$.

Thus Eq. (12) gives the "shape" of the spectral line.

2. GENERAL INVESTIGATION OF THE SHAPE OF THE SPECTRAL LINE

Let the frequency fluctuation $\Delta\omega_t$ be characterized by its correlation time τ_0, defined so that for $\tau > \tau_0$ the quantities $\Delta\omega_t$ and $\Delta\omega_{t+\tau}$ may be treated as practically independent.

Let us first consider the limiting case when the frequency fluctuations are slow or large, i.e., the case when $\overline{\Delta\omega^2}\,\tau_0^2 \gg 1$. To find the spectral density for this case, we consider the expression

$$\int_0^\tau (\tau-\xi)\,R(\xi)\, d\xi.$$

If τ is taken outside the integral, then after changing variables, we get

$$\int_0^\tau (\tau-\xi)R(\xi)\,d\xi=\tau^2\int_0^1(1-\xi)R(\tau\xi)\,d\xi.$$

In this case the spectral density (12) is given by

$$S(\omega)=\frac{A^2}{2\pi}\int_0^\infty \cos(\omega_0-\omega)\tau \tag{13}$$
$$\times\exp\Big[-\overline{\Delta\omega^2}\,\tau^2\int_0^1(1-\xi)R(\tau\xi)\,d\xi\Big]\,d\tau.$$

It can be shown that for $\overline{\Delta\omega^2}\,\tau_0^2\gg 1$, the quantity $R(\tau\xi)$ in Eq. (13) can be set equal to unity without significant error. As a result, we get

$$S(\omega)=\frac{A^2}{2\pi}\int_0^\infty \cos(\omega_0-\omega)\tau\exp\Big[-\frac{1}{2}\overline{\Delta\omega^2}\,\tau^2\Big]\,d\tau.$$

Calculating this integral, we get finally,

$$S(\omega)=\frac{A^2}{2}\left(2\pi\,\overline{\Delta\omega^2}\right)^{1/2}\exp\left\{-\frac{(\omega_0-\omega)^2}{2\,\overline{\Delta\omega^2}}\right\}. \tag{14}$$

Thus if there are slow or large fluctuations of the frequency, so that $\overline{\Delta\omega^2}\,\tau_0^2\gg 1$, the line is broadened and coincides in shape with a Doppler broadened line, with "line width" equal to $2\,(2\overline{\Delta\omega^2})^{1/2}$.

We now consider a second limiting case, opposite to the first. Suppose that there are rapid or small fluctuations of the oscillation frequency, so that $\overline{\Delta\omega^2}\,\tau_0^2\ll 1$. To find $S(\omega)$, we introduce the auxiliary quantity $k=k(\tau_0)$, such that

$$k(\tau_0)\int_{-\infty}^{+\infty}R(\tau)\,d\tau=1 \tag{15}$$

for all values of τ_0. Since for $\tau_0\to 0$ the quantity $R(\tau)$ differs from zero only for small values of τ [we recall that $R(0)=1$], it is obvious that

$$\lim_{\tau_0\to 0}k(\tau_0)R(\tau)=\delta(\tau),$$

where $\delta(\tau)$ is the delta function.

We write (13) in the form

$$S(\omega)=\frac{A^2}{2\pi}\int_0^\infty\cos(\omega_0-\omega)\tau$$
$$\times\exp\Big[-\frac{\overline{\Delta\omega^2}}{k}\tau^2\int_0^1(1-\xi)kR(\tau\xi)\,d\xi\Big]\,d\tau.$$

One can show that for $\overline{\Delta\omega^2}\,\tau_0^2\ll 1$, the quantity $kR(\tau\xi)$ in this expression can be set equal to $\delta(\tau\xi)$ without serious error, so that

$$S(\omega)=\frac{A^2}{2\pi}\int_0^\infty\cos(\omega_0-\omega)\,\tau\exp\Big[-\frac{\overline{\Delta\omega^2}}{2k}\tau\Big]\,d\tau.$$

Computing this integral, we get finally

$$S(\omega)=\frac{A^2}{2\pi}\,\frac{\overline{\Delta\omega^2}/2k}{(\overline{\Delta\omega^2}/2k)^2+(\omega_0-\omega)^2}. \tag{16}$$

This spectral density coincides in shape with that obtained by Bershtein for the natural broadening of the line.

Thus if there are rapid or small frequency fluctuations, so that $\overline{\Delta\omega^2}\,\tau_0^2\ll 1$, the line emitted by the oscillator suffers a broadening, and has a line shape identical with that for natural broadening with a "line width" equal to $\overline{\Delta\omega^2}/k$.

We note that from Eqs. (14) and (16) it follows that the Doppler and natural broadening may, in general, represent different special cases of the same process of "smearing" of the line emitted by an oscillator.

3. EXAMPLE

We consider the special case where the fluctuation of the generator frequency has a correlation coefficient equal to

$$R(\tau)=e^{-\alpha|\tau|},\quad \tau_0=1/\alpha.$$

The first limiting case, where $\overline{\Delta\omega^2}\gg\alpha^2$, leads without change to Eq. (14). To consider the second limiting case, where $\overline{\Delta\omega^2}\ll\alpha^2$, we determined the auxiliary quantity $k(\tau_0)$. From the definition (15) we find that $k(\tau_0)=\alpha/2$. Thus, for the second limiting case,

$$S(\omega)=\frac{A^2}{2\pi}\,\frac{\overline{\Delta\omega^2}/\alpha}{(\overline{\Delta\omega^2}/\alpha)^2+(\omega_0-\omega)^2}. \tag{17}$$

Now suppose that there is some arbitrary ratio between $\overline{\Delta\omega^2}$ and α^2. Returning to Eq. (12) and computing the integral in the argument of the exponential, we get:

$$S(\omega)=\frac{A^2}{2\pi}e^C\int_0^\infty\cos(\omega_0-\omega)\tau\exp\left[-B\tau-Ce^{-\alpha\tau}\right]d\tau,$$

where $C=\overline{\Delta\omega^2}/\alpha^2$, $B=\alpha C$. Computing this integral by expanding in series, we get the general expression for the spectral density for our example in the form

* The validity of the diffusion law for the phase fluctuations of a tube generator for arbitrary τ is also shown by the work of Rytov[8,9]

$$S(\omega)=\frac{A^2}{2\pi}e^{\overline{\Delta\omega^2}/\alpha^2} \tag{18}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{\overline{\Delta\omega^2}}{\alpha^2}\right)^m\frac{(\overline{\Delta\omega^2}/\alpha)+\alpha m}{[(\overline{\Delta\omega^2}/\alpha)+\alpha m]^2+(\omega_0-\omega)^2}.$$

If $\overline{\Delta\omega^2}\ll\alpha^2$, then only the first terms in the series (18) are important:

$$S(\omega)=\frac{A^2}{2\pi}\left[\frac{\overline{\Delta\omega^2}/\alpha}{(\overline{\Delta\omega^2}/\alpha)^2+(\omega_0-\omega)^2}-\frac{\overline{\Delta\omega^2}/\alpha}{\alpha^2+(\omega_0-\omega)^2}+\ldots\right]$$

where, as expected, the first term of the series coincides with Eq. (17).

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Vibration Spectrum of Disordered Crystal Lattices

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The vibration spectrum of a disordered crystal is studied. Calculations are made for an isotopic mixture, although the method could be applied under more general conditions. The mass of each atom is taken to be a random variable, and the deviation of the mass from its average value is not assumed to be small. The spectral density and the vibrational part of the free energy of the mixed crystal are determined.

THE determination of the vibration spectrum of a disordered system, for example a mixed solid, is a highly interesting problem. A similar problem was considered by one of the authors[1] in connection with the optical properties of mixed solids; at that time we investigated thoroughly only those aspects of the problem which are directly relevant to infra-red spectroscopy, (in particular the question of the existence of impurity frequencies). Dyson[2] considered the same problem for the special case of a disordered linear chain with nearest-neighbor interactions. But Dyson's method is by its very nature not capable of extension to three-dimensional systems.

In the present paper we describe a method which is free from these limitations. The method is an extension of earlier work by one of us.[3-6] We apply the method here to the case in which the atoms in the system differ only in mass (a mixture of isotopes). For the sake of clarity and simplicity of exposition, we consider only an idealized lattice in which all vibrations take place in one direction. This shortens the analysis considerably, without changing the essential nature of the problem.

There exists a deep-lying similarity in the effects of the destruction of translational invariance upon the energy spectra of phonons and of electrons. Hence the results of this investigation should be qualitatively valid also for electronic spectra.

1. THE METHOD OF TRACES

The equation for the vibrations of a lattice composed of a mixture of isotopes of a single element has the form

$$\sum_{\mathbf{r}'}\frac{A_{\mathbf{r}-\mathbf{r}'}}{m_{\mathbf{r}}}\chi(\mathbf{r}')-\omega^2\chi(\mathbf{r})=0. \tag{1}$$