Generalized Form of the Diffusion Equation for a Single Particle

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A general formulation is given of the diffusion equation for a single particle, from which follow a number of well known equations as special cases. By way of application, the scattering of particles in a homogeneous absorber is considered.

1. A general formulation of the diffusion equation for a single particle is given in the present paper. Other well known equations follow from it as special cases. The method applied in the present work is not here compared with other methods described in the literature; such a comparison will be given in a paper written by the author in collaboration with L. Pal and A. Bekesy. At the end of the paper, an application of the equation which represents practical interest will be examined.

The process of diffusion can be described by the function $\varphi(\mathbf{A}, \mathbf{U}, t)$, where $\mathbf{A} = A_1, A_2, \ldots, A_k$ are parameters which determine the state of the particle; in this case we do not limit ourselves only to such parameters which are strictly necessary for the determination of the state of the particle, but take into consideration other parameters, which depend in whole or in part on the rest; such parameters are included in order to determine their distribution. The vector U, which has the components $\mathbf{U} = U_1, U_2, \ldots$, defines the intervals of the components of A, and therefore, for each given state A, we can determine whether it lies within U or not. Finally, t denotes the time and φ the probability that a particle, which is in the state A at the time $t' = 0$, is transferred at time $t' = t$ into such a state A′ that

$$\mathbf{A}' \text{ lies within U.} \tag{1}$$

The choice of intervals U has no importance for the solution of the problem that has been posed. For many purposes it is appropriate to choose some or even all of the components of U to be infinitely narrow. In the latter case we consider φ as the probability *density* relative to those components for which the interval is bounded in the fashion mentioned. In particular, if all the components of U are narrow, we have, in place of Eq. (1),

$$\mathbf{A}' = \mathbf{U}. \tag{2}$$

2. We choose the time as an independent variable, since any other choice of independent variable leads to difficulties, about which we shall say something below. The quantity φ must satisfy the following initial condition:

$$\varphi(\mathbf{A}, \mathbf{U}, t = 0) = \Delta(\mathbf{A}, \mathbf{U}), \tag{3}$$

where

$$\Delta(\mathbf{A}, \mathbf{U}) = \begin{cases} 1, & \text{if A is within U} \\ 0 & \text{in any other case.} \end{cases}$$

In particular, if all the components of U lie in narrow intervals, then

$$\varphi(\mathbf{A}, \mathbf{U}, t = 0) = \delta(\mathbf{A} - \mathbf{U}). \tag{4}$$

If we had not chosen the time as the independent variable, but, say, the x coordinate of the particle, then, in place of an *initial* condition of type (3) or (4), it would have been necessary to furnish a *boundary* condition for x. It is not possible to give such a boundary condition arbitrarily; thus, for example, the boundary condition of the type

$$\varphi(\mathbf{A}, \mathbf{U}, x = 0) = \Delta(\mathbf{A}, \mathbf{U}) \tag{5}$$

leads to a contradiction. If the particle begins to move in the positive direction along the x axis, beginning from the point $x = 0$, then it can in general get back to the plane $x = 0$, while upon its return it can be shown to be outside of the interval U. The probability of such a case is always finite, although in many cases it can be small. Thus a condition of type (5) can be imposed only for a single instant of time; hence it follows that it is not possible to lay down (5) as a boundary condition for an independent variable x.

Although, strictly speaking, it is not possible to choose x as the independent variable, nevertheless in a series of cases, such a choice can be shown to be a suitable approximation. In these cases, we

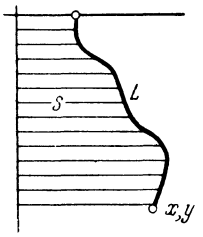

FIG. 1

neglect the errors which result from a possible rotation of the particle in the y,z plane from which it set out originally.

It is also possible to consider the length of path l as an independent variable. The difficulties which arise in this case are less serious than those in the choice of x as the independent variable; nevertheless, even in this case there remains the possibility that the velocity of the particle will finally reduce to zero. This leads to the appearance of singularities if l is chosen as the independent variable.

3. We consider the nature of the parameters **A**, while particular attention will be given to continuously and discontinuously varying quantities.

1) As components of A we can consider such quantities as the energy of the particle E, the momentum components p_x, p_y, p_z, the velocity components, etc. Such quantities change their values only in collisions (in each case, in the absence of an external field). Between collisions, these quantities remain fixed. In particular, it should be noted that the energy is often considered not as a continuously changing quantity rather than as a discontinuously changing one . It should be emphasized that such a treatment is of necessity approximate; ionizing losses are the result of many collisions, and in each collision, only a small part of the energy is lost. Thus the energy losses are discrete, and therefore we shall consider this process as discrete. Later it will be seen that, if we neglect certain small terms in our exact formula, we can obtain approximate formulas which result in the consideration of ionization as a continuous process.

2). We shall consider the components of **A** which change in a strictly continuous way. As such quantities, we have, for example, the coordinates x, y, z of the particle and the length l. An important quantity of this type is the area S between the trajectory of the particle and its projection on the x axis (Fig. 1). The latter quantities are not changed in a collision process, but change continuously in the interval between collisions. However, the derivatives of these quantities change discontinuously at the time of a collision. Thus, let

$A_1 = v_x$, $A_2 = v_y$, $A_3 = x$, $A_4 = y$, $A_5 = S$ (v_x, v_y are velocity components); then, between collisions, $A_1(t) = A_1(0)$, $A_2(t) = A_2(0)$, $A_3(t) = A_3(0) + A_1(0)t$, $A_4(t) = A_4(0) + A_2(0)t$, $A_5(t) = A_5(0) + A_3(0)t + {}^1/_2 A_1(0)A_2(0)t^2$ (for $0 \leqslant t < t_1$), where $A_k(t)$ is the value of the component A_k at $t \geq 0$, and under the condition that up to $t_1 > t$ no collision occurred.

3). If external forces act on the particle, then such parameters as the energy and momentum will change continuously between collisions, and discontinuously at collisions. A case of practical importance is the case of the deflection of a particle under the combined action of a magnetic field and diffusion.

In order not to introduce any limitations, we assume that, in general,

$$A_k(t) = A_k^{(0)}(0) + A_k^{(1)}(0)\,t + {}^1/_2 A_k^{(2)}(0)\,t^2 + \ldots$$

and consider all variables $A_k^{(l)}$ as stochastic variables which can change or not change upon collisions. However, we assume that a finite (and generally speaking, small) number of derivatives of $A_k^{(l)}$ will give a sufficiently accurate approximation*.

In what follows, when the symbol **A** is used, it will be assumed that all components $A_k^{(l)}$ are included in it, and correspondingly, that the vector **U** includes all components which relate to all the $A_k^{(l)}$. The quantities A thus introduced are certainly not all independent of each other; below, we shall have to deal with the question of the inclusion of certain of them in the course of the computation, but initially, we shall not take into account the possibility of the existence of a dependence between the quantities under consideration.

4). Variables which change continuously in accordance with the laws of probability will not be considered. Such quantities are not met with in practice and their introduction would provide an unnecessary complication. Those quantities which arbitrarily change "almost" continuously (for example, due to ionization losses), can be successfully considered as discontinuously changing quantities.

4. In order to write down the diffusion equation for φ it is necessary to introduce the collision cross section. We define

$$w(\mathbf{A}(t), \mathbf{A}'')\, d\mathbf{A}''\, dt \qquad (5a)$$

* The method developed here has been generalized by L. Pal; he has deduced the diffusion equation for cases in which $\mathbf{A}(t)$ is an arbitrary function of time, while not making use of the power series expansion.

as the probability that a particle, whose state at time t is described by the vector $\mathbf{A}(t)$, undergoes such a collision in the time interval t, $t + dt$ that after the collision the particle is found in the interval $\mathbf{A}''$, $\mathbf{A}'' + d\mathbf{A}''$. It is obvious that

$$d\mathbf{A}'' = \prod_{k,l} dA_k^{(l)''}.$$

In place of A we write $\mathbf{A}(t)$, since the state of the particle changes even in the absence of a collision. Use of the vector $\mathbf{A}(t)$ denotes that all computed values of all essential time derivatives $d^l A_k(t)/dt^l$; are considered; this is important because the instantaneous value of only one of the parameters does not sufficiently determine the state of the particle.

Some of the parameters $A_k^{(l)}$ do not change abruptly in collisions. We denote those quantities which do change abruptly by **a**, and those which do not so change by **b**. Thus $\mathbf{A} = \mathbf{a}, \mathbf{b}$; we can separate $w(\mathbf{A}(t), \mathbf{A}'')$ in the following manner:

$$w(\mathbf{A}(t), \mathbf{A}'') = \bar{w}(\mathbf{A}(t), \mathbf{a}'')\,\delta(\mathbf{b}(t) - \mathbf{b}''(t)), \quad (6)$$

where the δ function is the product of δ functions for each of the components b.

The function $\bar{w}$ can depend on the value of $\mathbf{b}(t)$ as shown above, but the function $\mathbf{b}(t)$ cannot change in collisions. An example of such a case is scattering in an inhomogeneous absorber. In this case, the cross section depends on the coordinates, in spite of the fact that the coordinates do not change in the collision. It would have been possible to write down the diffusion equation in terms of a regular $\bar{w}$ function, and thus avoid the application of δ functions. However, it is rather complicated to divide **A** into the groups **a** and **b**; therefore we prefer to employ w and the δ function. In order to avoid difficulties connected with the application of the δ function, we can replace the actual δ function by regular functions, e.g., by the Gaussian function

$$\delta(x) = (2\pi\varepsilon)^{-1/2} \exp(-x^2/2\varepsilon), \quad \varepsilon > 0, \quad (7)$$

where ϵ is small but different from zero. Substituting Eq. (7) in Eq. (6), we introduce an error which can be made sufficiently small if ϵ is chosen sufficiently small. In fact, let us consider as an example, the x component of a mixture. In this case, the introduction of Eq. (7) in place of the actual δ function is equivalent to the supposition that in each collision x changes discontinuously by an amount of order ϵ. If, for example, $\epsilon \sim 10^{-100}$ cm, then such a supposition would undoubtedly have been allowable, since it does not lead to any noticeable effect. The mathematical consequence of such a supposition, which does not have any physical consequence, is the possibility of considering $w(\mathbf{A}, \mathbf{A}'')$ as a regular function.

5. We can now write down the diffusion equation for $w(\mathbf{A}, \mathbf{U}, t)$. We have

$$\varphi(\mathbf{A}, \mathbf{U}, t) = \varphi_0(\mathbf{A}, \mathbf{U}, t) + \varphi_1(\mathbf{A}, \mathbf{U}, t), \quad (8)$$

where φ_0 is the probability that the final state within U arises without a collision, and φ_1 is the probability that the final state within U arises as the result of one or more collisions.

The probability of an absence of a collision in the time interval from 0 to t' can be written as $\varphi_0(\mathbf{A}, t')$; the probability of an absence of a collision in the interval between t' and t is expressed as $\varphi(\mathbf{A}(t'), t - t')$. Thus the probability that no collision takes place during the entire interval is equal to

$$\varphi_0(\mathbf{A}, t) = \varphi_0(\mathbf{A}(0), t')\,\varphi_0(\mathbf{A}(t'), t - t'). \quad (9)$$

The general solution of Eq. (9) is

$$\varphi_0(\mathbf{A}, t) = \exp\left\{-\int_0^t w(\mathbf{A}(t''))dt''\right\}, \quad (10)$$

which can be established by substituting Eq. (10) in Eq. (9), where

$$w(\mathbf{A}(t'')) = \int_{\mathbf{A}''} w(\mathbf{A}(t''), \mathbf{A}'')\,d\mathbf{A}'' \quad (11)$$

and integration is carried out over all components of $\mathbf{A}''$. Equation (11) is so chosen that the entire probability of collision is equal to the value given by Eq. (5a).

Thus the first term on the right side of Eq. (8) is determined by the expression

$$\varphi_0(\mathbf{A}, \mathbf{U}, t) = \varphi_0(\mathbf{A}, t)\,\Delta(\mathbf{A}(t), \mathbf{U}). \quad (12)$$

The expression on the right side of Eq. (12) is equal to the product of the probability of the absence of a collision between 0 and t and the probability that a continuously changing **A** undergoes a transition into the interval U or remains there.

In order to determine $\varphi_1(\mathbf{A}, \mathbf{U}, t)$, we consider a case in which the first collision takes place in the interval $t - t'$, $t - t'' + dt''$. The probability of such an event is

$$\varphi_0(\mathbf{A}, t - t'')\,w(\mathbf{A}(t - t''), \mathbf{A}'')\,d\mathbf{A}''dt'',$$

where it is assumed that the first collision is such that the state of the particle is found within the interval $\mathbf{A}''$, $\mathbf{A}'' + d\mathbf{A}''$. The probability of

the passage of a particle into the interval U after a time t'' is $\varphi(\mathbf{A}'', \mathbf{U}, t'')$, and therefore the total probability that, as a result of one or more collisions, the particle is found in the interval U, is equal to

$$\varphi_1(\mathbf{A}, \mathbf{U}, t) = \int_0^t \int_{\mathbf{A}''} \varphi_0(\mathbf{A}, t - t'')\, w(\mathbf{A}(t - t''), \mathbf{A}'')\, \varphi(\mathbf{A}'', \mathbf{U}, t'')\, d\mathbf{A}'' dt''. \tag{13}$$

Combining φ_0 and φ_1, we then obtain

$$\varphi(\mathbf{A}, \mathbf{U}, t) = \varphi_0(\mathbf{A}, t)\, \Delta(\mathbf{A}(t), \mathbf{U}) + \int_0^t \int_{\mathbf{A}''} \varphi_0(\mathbf{A}, t - t'')\, w(\mathbf{A}(t - t''), \mathbf{A}'')\, \varphi(\mathbf{A}'', \mathbf{U}, t'')\, d\mathbf{A}''\, dt''. \tag{14}$$

Equation (14), together with Eq. (12), gives the general diffusion equation. For practical purposes, Eq. (14) can be transformed to a more suitable form.

6. We transform Eq. (14) into an integro-differential equation in the following way. Setting $t' = t - t''$, we get from Eq. (9)

$$\varphi_0(\mathbf{A}, t - t'') = \varphi_0(\mathbf{A}, t) / \varphi_0(\mathbf{A}(t - t''), t'').$$

Introducing this expression in Eq. (14), we obtain

$$\varphi(\mathbf{A}, \mathbf{U}, t) = \varphi_0(\mathbf{A}, t) \left\{ \Delta(\mathbf{A}(t), \mathbf{U}) + \int_0^t \int_{\mathbf{A}''} \frac{w(\mathbf{A}(t - t''), \mathbf{A}'')}{\varphi_0(\mathbf{A}(t - t''), t'')} \varphi(\mathbf{A}'', \mathbf{U}, t'')\, d\mathbf{A}'' dt'' \right\}.$$

Differentiating this expression with respect to t, we have

$$\frac{\partial \varphi(\mathbf{A}, \mathbf{U}, t)}{\partial t} = \frac{\partial \ln \varphi_0(\mathbf{A}, t)}{\partial t} \varphi(\mathbf{A}, \mathbf{U}, t) + \int_{\mathbf{A}''} \omega(\mathbf{A}, \mathbf{A}'')\, \varphi(\mathbf{A}'', \mathbf{U}, t)\, d\mathbf{A}'' + \left(\frac{\partial I(\tau)}{\partial \tau} \right)_{\tau=0} \varphi_0(\mathbf{A}, t), \tag{15}$$

where

$$I(\tau) = \Delta(\mathbf{A}(t + \tau), \mathbf{U}) + \int_0^t \int_{\mathbf{A}''} \frac{w(\mathbf{A}(t - t'' + \tau), \mathbf{A}'')}{\varphi_0(\mathbf{A}(t - t'' + \tau), t'')} \varphi(\mathbf{A}'', \mathbf{U}'', t'')\, d\mathbf{A}'' dt'',$$

and finally

$$I(0) = \varphi(\mathbf{A}, \mathbf{U}, t) / \varphi_0(\mathbf{A}, t).$$

In order to carry out the differentiation, we recall that

$$\frac{\partial}{\partial \tau} f(\mathbf{A}(t + \tau - t'')) = \frac{\partial}{\partial \tau} f[A_1^{(0)} + A_1^{(1)}(t + \tau - t'') + {}^1/_2 A_1^{(2)}(t + \tau - t'')^2 \ldots \ldots A_2^{(0)} + A_2^{(1)}(t + \tau - t'') + {}^1/_2 A_2^{(2)}(t + \tau - t'')^2 + \ldots]$$

and thus,

$$\left(\frac{\partial}{\partial \tau} f(\mathbf{A}(t + \tau - t'')) \right)_{\tau=0} = \sum_{k,l} A_k^{(l+1)} \frac{\partial f(\mathbf{A}(t - t''))}{\partial A_k^{(l)}};$$

consequently,

$$\left(\frac{\partial I(\tau)}{\partial \tau} \right)_{\tau=0} = \sum_{k,l} A_k^{(l+1)} \frac{\partial I(0)}{\partial A_k^{(l)}} + \sum_{k,l} A_k^{(l+1)} \frac{\partial}{\partial A_k^{(l)}} \left(\frac{\varphi(\mathbf{A}, \mathbf{U}, t)}{\varphi_0(\mathbf{A}, t)} \right).$$

Substituting in Eq. (15) we get

$$\frac{\partial \varphi(\mathbf{A}, \mathbf{U}, t)}{\partial t} = \int_{\mathbf{A}''} w(\mathbf{A}, \mathbf{A}'')\, \varphi(\mathbf{A}'', \mathbf{U}, t)\, d\mathbf{A}'' + \sum_{k,l} A_k^{(l+1)} \frac{\partial \varphi(\mathbf{A}, \mathbf{U}, t)}{\partial A_k^{(l)}} + \left(\frac{\partial}{\partial t} - \sum_{k,t} A_k^{(l+1)} \frac{\partial}{\partial A_k^{(l)}} \right) \ln \varphi_0(\mathbf{A}, t)\, \varphi(\mathbf{A}, \mathbf{U}, t).$$

It is easy to see that the last term is equal to $-w(\mathbf{A})\, \varphi(\mathbf{A}, \mathbf{U}, t)$. Finally, the diffusion equation takes the form

$$\left(\frac{\partial}{\partial t} + w(A) - \sum_{k,l} A_k^{(l+1)} \frac{\partial}{\partial A_k^{(l)}} \right) \varphi(\mathbf{A}, \mathbf{U}, t) = \int_{\mathbf{A}''} w(\mathbf{A}, \mathbf{A}'')\, \varphi(\mathbf{A}'', \mathbf{U}, t)\, d\mathbf{A}'' \ldots \tag{16}$$

7. As an example of an application of our formula, we consider the scattering of a particle in a homogeneous absorber and, in particular, the distribution of the area between the projection of the trajectory and a straight line (see Fig. 2). Knowledge of the distribution of this quantity can be useful in the determination of the energy of the

particle from its track in the emulsion.

Let us consider a particle which moves at a small angle to the x axis, and only those cases of scattering which do not lead to large departures from this direction. We neglect quantities of order A^2, where A is the projection of the angle of the track on the x, z plane.

Neglecting large angle scattering, we can also neglect the possibility of recurrence of the initial value of x for the moving particle, and therefore we can use x as the independent variable instead of t (see Sec. 2).

The particle is thus characterized by its energy E, the coordinates x, Z, the angle A with the x axis and the area between the trajectory and the x axis (Fig. 2). Physical interest attaches not to S but to the area of the segment s, i.e., (see Fig. 2)

$$s = S - {}^1/_2 (Z + Z') x. \tag{17}$$

It would be more suitable, however, first to determine the distribution of S and then derive the distribution of s.

In order that the notation be the same as used in Eq. (16), we write

$$E = A_1, \quad A_1(x) = A_1^{(0)},$$
$$A = A_2, \quad A_2(x) = A_2^{(0)},$$
$$Z = A_3, \quad A_3(x) = A_3^{(0)} + xA_3^{(1)},$$
$$\text{where} \quad A_3^{(1)} = A_2^{(0)} = A,$$
$$S = A_4, \quad A_4(x) = A_4^{(0)} + xA_4^{(1)} + {}^1/_2 x^2 A_4^{(2)},$$

where $A_4^{(1)} = A_3^{(0)} = Z$, $A_4^{(2)} = A_2^{(0)} = A$. The diffusion equation then has the form

$$\left(\frac{\partial}{\partial x} + w(\mathbf{A}) - A_3^{(1)} \frac{\partial}{\partial A_3^{(0)}} - A_4^{(1)} \frac{\partial}{\partial A_4^{(0)}} - A_4^{(2)} \frac{\partial}{\partial A_4^{(1)}}\right) \varphi(\mathbf{A}, \mathbf{U}, x) = \int_{\mathbf{A}''} w(\mathbf{A}, \mathbf{A}'') \varphi(\mathbf{A}'', \mathbf{U}, x)\, d\mathbf{A}'' \tag{18}$$

with the additional conditions

$$A_4^{(2)} = A_3^{(1)} = A_2^{(0)} = A, \tag{19}$$
$$A_4^{(1)} = A_3^{(0)} = Z. \tag{20}$$

Equation (19) can be substituted directly in Eq. (18), since Eq. (18) does not contain any derivative with respect to $A_4^{(2)}$ or $A_3^{(1)}$. However, direct substitution of Eq. (20) in Eq. (18) is not possible, since the derivatives with respect to $A_4^{(1)}$ and $A_3^{(0)}$ apply to different processes, even though the computed values of these quantities can coincide. We demonstrate that we can eliminate Eq. (20). Let $A_4^{(1)} = Z_1$, $A_3^{(0)} = Z$; then we have in place of Eq. (18):

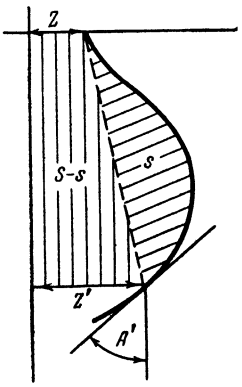

FIG. 2

$$\left(\frac{\partial}{\partial x} + w(\mathbf{A}) - A\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial S} - A\frac{\partial}{\partial Z_1}\right) \times \varphi(\mathbf{A}, \mathbf{U}, x) = \int_{\mathbf{A}''} w(\mathbf{A}, \mathbf{A}'') \varphi(\mathbf{A}'', \mathbf{U}, x)\, d\mathbf{A}''.$$

We now set

$$\varphi(\mathbf{A}, \mathbf{U}, x) = \bar{\varphi}(\mathbf{A}, \mathbf{U}, x) f(Z - Z_1), \tag{21}$$

where

$$\partial\bar{\varphi} / \partial z_1 = 0, \quad f(Z - Z_1) \neq 0.$$

Substituting Eq. (21) in Eq. (20) and dividing by $f(Z - Z_1)$, we get (under the assumption that $w(\mathbf{A}, \mathbf{A}'')$ clearly does not depend on Z or Z_1):

$$\left(\frac{\partial}{\partial x} + w(\mathbf{A}) - A\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial S}\right) \bar{\varphi}(\mathbf{A}, \mathbf{U}, x) = \int_{\mathbf{A}''} w(\mathbf{A}, \mathbf{A}'') \bar{\varphi}(\mathbf{A}'', \mathbf{U}, x)\, d\mathbf{A}'', \tag{22}$$

for $Z = Z_1$ we get $\varphi(\mathbf{A}, \mathbf{U}, x) = f(0)\, \bar{\varphi}(\mathbf{A}, \mathbf{U}, x)$.

We can assume $f(0) = 1$ and, consequently, in the most interesting region, $\varphi = \bar{\varphi}$. For this reason, we omit the bar over φ in Eq. (22). We can therefore remove completely both the condition (19) and condition (20), and solve Eq. (22) without additional conditions.

8. In what follows we introduce approximations for the purpose of simplifying Eq. (22). Thus, we assume that $w(\mathbf{A}, \mathbf{A}'')$ depends only on the energy and the absolute value of the angle of scattering. We then write

$$w(\mathbf{A}, \mathbf{A}'') = \bar{w}(E; E'', |A - A''|) \times \delta(Z - Z'')\,\delta(S - S'');$$

we then get, in place of Eq. (22),

$$\left(\frac{\partial}{\partial x} - A\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial S}\right)\varphi(E, A, Z, S; \mathbf{U}; x) = \int\int \bar{w}(E; E'', |A - A''|)\{\varphi(E'', A'', Z, S; \mathbf{U}; x) - \varphi(E, A, Z, S; \mathbf{U}; x)\}\,dE''dA''.$$

If we assume further that, for $E - E'' \approx 0$ and $A - A'' \approx 0$, the quantity $\bar{w}$ has a sharp maximum, i.e., that both the energy losses and the scattering essentially consist of many components of small value, then the expression under the integral will be

$$\varphi(E'', A'', Z, S; \mathbf{U}; x) - \varphi(E, A, Z, S; \mathbf{U}; x) = \left\{(E'' - E)\frac{\partial}{\partial E} + (A'' - A)\frac{\partial}{\partial A} + \frac{1}{2}(E'' - E)^2\frac{\partial^2}{\partial E^2} + (E'' - E)(A'' - A)\frac{\partial^2}{\partial E\,\partial A} + \frac{1}{2}(A'' - A)^2\frac{\partial^2}{\partial A^2} + \cdots\right\}\varphi(E, A, Z, S; \mathbf{U}; x).$$

Upon integration, all terms proportional to $A\text{-}A''$ vanish because of symmetry and there is left

$$\int\int \cdots dE''dA'' = -a(E)\frac{\partial}{\partial E} + \frac{1}{2}b(E)\frac{\partial^2}{\partial E^2} + \frac{1}{2}\sigma(E)\frac{\partial^2}{\partial A^2} + \ldots,$$

where

$$a(E) = \int (E - E'')\,w(E, E'')\,dE'',$$

$$w(E, E'') = \int w(E, E'', |A - A''|)\,dA'',$$

$$b(E) = \int (E - E'')^2\,w(E, E'')\,dE'',$$

$$\sigma(E) = \int\int (A - A'')\,w(E, E''|A - A''|)\,dA''dE''.$$

In the latter approximation we have, consequently,

$$\left(\frac{\partial}{\partial x} + a(E)\frac{\partial}{\partial E} - \frac{1}{2}b(E)\frac{\partial^2}{\partial E^2} + \frac{1}{2}\sigma(E)\frac{\partial^2}{\partial A^2} - A\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial S}\right)\varphi(E, A, Z, S; \mathbf{U}; x) = 0. \tag{23}$$

Further, we neglect the fluctuations of energy loss, i.e., we set $b(E) \sim 0$. Finally, for convenience in later calculations, we introduce more symmetrical notation, namely, $E = X_0$, $A = X_1$, $Z = X_2$, $S = X_3$ and $\mathbf{X} = X_0, X_1, X_2, X_3$; furthermore, we shall assume that U denotes a narrow interval in the neighborhood of

$$\mathbf{X}' = X_0', X_1', X_2', X_3'.$$

Thus our problem reduces to the solution of the equation

$$\left(\frac{\partial}{\partial x} + a(X_0)\frac{\partial}{\partial X_0} + \frac{1}{2}\sigma(X_0)\frac{\partial^2}{\partial X_1^2} - X_1\frac{\partial}{\partial X_3}\right)\varphi(\mathbf{X}, \mathbf{X}', x) \tag{24}$$

under the initial condition

$$\varphi(\mathbf{X}, \mathbf{X}', 0) = \delta(\mathbf{X} - \mathbf{X}') = \delta(X_0 - X_0')\,\delta(X_1 - X_1') \times \delta(X_2 - X_2')\,\delta(\mathbf{X}_3 - \mathbf{X}_3'). \tag{25}$$

9. For the solution of Eq. (24) we separate that part of the function which depends on the energy, i.e., we set

$$\varphi(\mathbf{X}, \mathbf{X}', x) = f(X_0, x)\,\bar{\varphi}(\mathbf{X}, x). \tag{26}$$

Substituting Eq. (26) in Eq. (24), we get

$$\frac{\partial f(X_0)}{\partial x} + a(X_0)\frac{\partial f(X_0)}{\partial X_0} = 0.$$

The solution of the equation above, which satisfies the condition (25), has the following form

$$f(X_0, x) = \delta\left(x - \int_{X'}^{X_0}\frac{\partial X_0''}{a(X_0)}\right). \tag{27}$$

Substituting Eq. (26) in Eq. (24), we find that, for all values of the variables for which $f \neq 0$, the function φ must satisfy Eq. (24). It follows from Eq. (27) that $\bar{\varphi}$ must satisfy Eq. (24) only for

$$X_0 = X_0(x), \tag{28}$$

where

$$\int_{X_0'}^{X_0(x)}\frac{dX_0''}{a(X_0'')} = x.$$

Thus we can rewrite Eqs. (24) and (25) in the following form

$$\left(\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x)\frac{\partial^2}{\partial X_1^2} - X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_3}\right)\bar{\varphi}(X, x) = 0, \tag{29}$$

$$\sigma(x) \equiv \sigma(X_0(x)),$$
$$\bar{\varphi}(X, 0) = \delta(X_1 - X_1')\,\delta(X_2 - X_2')\,\delta(X_3 - X_3')$$

and it is now possible not to consider the components X_0.

10. To obtain a solution of Eq. (29) in clear form we apply a Laplace transformation relative to the three components of X to $\bar{\varphi}$ and Eq. (29). Consequently we introduce the function

$$\psi(\vec{\lambda}, x) \equiv \psi(\lambda_1, \lambda_2, \lambda_3, x) = \int_{-\infty}^{+\infty} e^{\vec{\lambda}X}\,\bar{\varphi}(\mathbf{X}, x)\,d\mathbf{X}, \tag{30}$$

where $\vec{\lambda}\mathbf{X} = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$;
$d\mathbf{X} = dX_1\,dX_2\,dX_3$.

After the transformation, Eq. (29) takes the form

$$\left(\frac{\partial}{\partial X} + \frac{1}{2}\sigma(x)\lambda_1^2 + \lambda_2\frac{\partial}{\partial\lambda_1} + \lambda_3\frac{\partial}{\partial\lambda_2}\right)\psi(\lambda, x) = 0, \tag{31}$$
$$\psi(\vec{\lambda}, 0) = \exp\{\vec{\lambda}\mathbf{X}'\}.$$

Assuming that (32)

$$\ln\psi(\vec{\lambda}, X) = \sum_{k=1}^{3}\alpha_k\lambda_k + \sum_{i,k=1}^{3}\beta_{ik}\lambda_i\lambda_k;\ \beta_{ik} = \beta_{ki},$$

we find from Eq. (31)

$$\alpha_1 = A',\ \alpha_2 = Z' - A'x, \tag{33}$$
$$\alpha_3 = S' - Z'x + {}^1/_2 A'x^2;$$
$$\beta_{11} = \sigma_0(x),\quad \beta_{12} = -\sigma_1(x),\quad \beta_{13} = {}^1/_2\sigma_2(x), \tag{34}$$
$$\beta_{22} = \sigma_2(x),\quad \beta_{23} = -{}^1/_2\sigma_3(x),\quad \beta_{33} = {}^1/_4\sigma_4(x),$$

where

$$\sigma_k(x) = \int_0^x (x - x')^k\sigma(x')\,dx'. \tag{35}$$

Substituting Eqs. (33)-(35) in Eq. (32), we obtain the Laplace transform of the solution of Eq. (29). Carrying out the inverse transformation, we find the solution in the following form:

$$\bar{\varphi}(\mathbf{X}, x) = (2\pi\det\beta_{ik})^{-1/2} \times \exp\left\{-\frac{1}{2}\sum_{i,k}\tilde{\beta}_{ik}(X_i - \alpha_i)(X_k - \alpha_k)\right\}, \tag{36}$$

where $\tilde{\beta}_{ik}$ is equal to the minor $\beta_{ik}/\det\beta_{ik}$, i.e., $\tilde{\beta}_{ik}$ are the terms of the inverse matrix for β_{ik} where $\sum_k \tilde{\beta}_{ik}\beta_{kl} = \delta_{ik}$.

11. Equation (36) gives a similar expression for the instantaneous distribution of A', Z', S' at a given depth x, for given initial values of A, Z, S. If we take the theoretical value of $\sigma(x)$, we can compute numerically the values of all the coefficients. By way of an example, we consider in more detail the properties of the distribution (36) under neglect of energy losses, i.e., considering $\sigma(x) =$ const. For simplicity, we take

$${}^1/_2\sigma(x) = 1. \tag{37}$$

$\sigma(x)$ has the dimensions of inverse length and therefore Eq. (37) determines the unit of length.

Substituting Eq. (37) in Eq. (35) we find $\sigma_k(x) = x^{k+1}/(k+1)$; further, from Eq. (34) it follows that

$$\beta_{11} = x,\quad \beta_{12} = -{}^1/_2 x^2,\quad \beta_{13} = {}^1/_6 x^3,$$
$$\beta_{22} = {}^1/_3 x^3,\quad \beta_{23} = -{}^1/_8 x^4,$$
$$\beta_{33} = {}^1/_{20} x^5,$$
$$\det\beta_{ik} = x^9/8640,\quad 8640 = 3!4!5!/2.$$

Substituting $\mathbf{X} = 0$ in Eq. (36), i.e., $A = Z = S = 0$, and considering that the particles set out parallel to the X axis from the origin of the coordinates, we find

$$\bar{\varphi}(0, \mathbf{X}', x) = \gamma x^{-9/2}\exp\left\{-\frac{1}{2x}\left(9A'^2 + 72A'\left(\frac{Z'}{x} - A'\right) + 120A'\left(\frac{S'}{x^2} - \frac{Z'}{x} + \frac{1}{2}A'\right) + 192\left(\frac{Z'}{x} - A'\right)^2 + 720\left(\frac{Z'}{x} - A'\right)\left(\frac{S'}{x^2} - \frac{Z'}{x} + \frac{1}{2}A'\right) + 720\left(\frac{S'}{x^2} - \frac{Z'}{x} + \frac{1}{2}A'\right)^2\right)\right\};$$
$$\gamma = (8640/2\pi)^{1/2}.$$

Writing, for brevity, $\varphi(\mathbf{X}', x)$ in place of $\bar{\varphi}(0, \mathbf{X}', x)$, we find, after transformation of the polynomial in the exponent,

$$\varphi(\mathbf{X}', x) = \gamma x^{-9/2} \exp\left\{-\frac{1}{2x}(9A'^2 - 72A'Z'/x + 120A'S'/x^2 + 192Z'^2/x^3 - 720Z'S'/x^3 + 720S'^2/x^4)\right\}. \quad (38)$$

Equation (38) gives in clear form the instantaneous distribution of A', Z', S'; with the help of this formula, it is possible to determine, in the usual fashion, the mean quantities of momentum, correlation coefficient, etc. From Eq. (38) we can also obtain the homogeneous distribution of A', Z' and $s' = S' - {}^1/_2 xZ'$; we get (see Fig. 2)

$$\varphi(A', Z', s'; x) = \gamma x^{-9/2} \times \exp\left\{-\frac{1}{2x}(9A'^2 - 12A'Z'/x + 12Z'^2/x + 120A's'/x^2 + 720s'^2/x^4)\right\}. \quad (39)$$

We can average over A' or Z' and, consequently, obtain the double distribution:

$$\varphi(A', s'; x) = \int_{-\infty}^{+\infty} \varphi(A', Z', s'; x)\, dZ' = \gamma_1 x^{-3} \exp\left\{-\frac{1}{2x}(4A'^2 + 120A's'/x^2 + 720s'^2/x^4)\right\}. \quad (40)$$

However, the distribution (40) is not of immediate interest. We are interested in the instantaneous distribution of the quantities $a' = A' - Z'/x$ and s'. The importance of these quantities is clear from Fig. 3; a' is the angle between the trajectory of the particle and the straight line connecting the origin of the trajectory with the final point.

The distribution for a', s' has the form

$$\varphi(a', s'; x) = \gamma_2 x^{-3} \times \exp\left\{-\frac{1}{2x}(8a'^2 + 80a's'/x^2 + 320s'^2/x^4)\right\}. \quad (41)$$

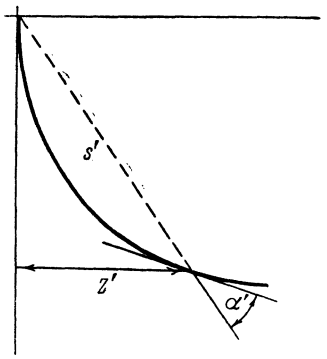

FIG. 3

As is evident from Eq. (41), there is a strong negative correlation between a' and s'; the reason for this negative correlation can be found in a consideration of Fig. 3. In particular, according to Eq. (41), for a fixed value of A', the most probable value of s' is equal to

$$s'_0(a') = -a'x^2/8. \quad (42)$$

Similarly, the most probable angle a', for a fixed value of s', is obtained from Eq. (41):

$$a'_0(s') = -5s'/x^2. \quad (43)$$

It is interesting to compare the values of the coefficients in Eqs. (42) and (43). Averaging over a' we can determine the distribution of s'. We obtain, finally,

$$\varphi(s'; x) = \gamma_3 x^{-5/2} \exp\{-60s'^2/x^5\}.$$

The distribution is strictly Gaussian, with the most probable value of $s' = 0$.

In the future, we hope to return to the question of the application of the formulas just developed to the problem of the measurement of the energy of particles which undergo scattering in a photographic emulsion.

Translated by R. T. Beyer.
61