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The Theory of the Interaction of an Electron Gas with the Vibrations of the Crystalline Lattice

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The method of the Green's function is applied to the problem of the interaction of electrons with the phonon field. The electron energy spectrum and electron momentum spectrum at absolute zero have been computed.

1. INTRODUCTION

THE question of the interaction of conduction electrons with the vibrations of the crystalline lattice has in recent years acquired considerable importance for a whole series of problems. It has been observed repeatedly (see, for example, references 1-3) that, within the framework of the phonon picture of the crystal, this problem is anologous to problems considered in contemporary quantum field theory (only with this difference, that the momentum of the phonon has, of course, an upper limit). Salam[3], with the help of the usual technique of Feynman diagrams formally constructed an S matrix for the electron-phonon field. However, the calculation of just one S matrix does not always present immediate interest. In a series of cases, there is physical interest in the energy spectrum of the electron which interacts with the vibrations of the lattice, and the state distribution functions of the electrons. These problems are most simply solved by the Green's function method, since the distribution function can be obtained directly from the latter.

In the present research, the connection is established between the Green's function and the distribution function, and the state of the system at absolute zero is investigated. For concreteness, we consider a system of electrons that are not directly interacting (an electron gas). However, it must be kept in mind that all considerations can, by a simple change of notation, carry over also to the case in which arbitrary elementary excitations of the Fermi type interact with phonons.

2. GREEN'S FUNCTION AND THE DISTRIBUTION FUNCTION

We denote by $\psi_s(x)$ and $A(x)$ the quantum wave functions of the electron and the phonon fields ($x = \{\mathbf{x}, x_0\}$, s = spin index; only longitudinal vibrations of the lattice are taken into consideration, because the interaction of electrons with the transverse vibrations can usually be neglected). The Lagrangian of the interaction with the acoustical vibrations has the form ($\hbar = c = 1$; c=velocity of sound):

$$L_{\rm int} = \{g\psi_s^*(x)\,\psi_s(x) + \rho(x)\}\,\varphi(x); \tag{2.1}$$

$$\varphi = \partial A/\partial x_0.$$

Here $\rho(x)$ is the "external charge density" (analogous to the "external current" in quantum electrodynamics), g = coupling constant (it can be expressed by the known constant C which enters into the theory of electrical conductivity[4]). The matrix S is given by the well known expression (we make use of the interaction representation)

$$S = T\{\exp(i\int L(x)\,dx)\}, \tag{2.2}$$

where the symbol $T\{\ldots\}$ denotes the T product. Naturally, S is an eigenfunctional of $\rho(x)$. The electron Green's function G and the "potential" $a(x)$ are obtained in the same way as in quantum electrodynamics (see, for example, reference 5):

$$G_{ss'}(x,\ y) = \frac{i}{\langle S_0\rangle}\langle T\{\psi_s(x)\,\psi^*_{s'}(y)S\}\rangle_0, \tag{2.3}$$

$$a(x) = -i\ \delta\ln\langle S\rangle_0/\delta\rho(x) \tag{2.4}$$

(the symbol $<\ldots>_0$ means "averaged over the ground state of the system in the absence of interaction"). The Green's function for phonons could also be introduced, but it is not necessary for us in the present research.

It is easy to see that the Green's function $G_{ss'}(x, y)$ directly describes the state distribution

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[1] H. Frölich, Phys. Rev. **79**, 845 (1950).

[2] S. B. Tiablikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **21**, 377 (1951).

[3] A. Salam, Progr. Theoret. Phys. **9**, 550 (1953).

[4] H. Bethe and A. Sommerfeld, *Electron Theory of Metals*.

of the electrons (taking into account the interaction with the phonons).

Thus

$$\lim_{\substack{x_0\to y_0=t\\ x_0<y_0}} G(x,\ y) = -\,iR_1^e(\mathrm{x},\ \mathrm{y};\ t); \tag{2.5}$$

$$\lim_{\substack{x_0\to y_0=t\\ x_0>y_0}} G(x,\ y) = iR_1^h(\mathrm{x},\ \mathrm{y};\ t),$$

where R_1^e and R_1^h are "single particle" density matrices for electrons and holes, respectively (calculated with consideration of the interaction with phonons). We now put $\psi(x)$ in the form

$$\psi_s(x) = \int d\lambda a_s(\lambda)\,\varphi_\lambda(x) \tag{2.6}$$

($a_s(\lambda)$ are the Fermi operators, which operate on the occupation numbers, $\varphi_\lambda(x)$ are the eigenfunctions of a certain additive operator, the eigenvalues of which are λ; the index λ can also denote the set of several quantities). Then (for $x_0 < y_0$):

$$G_{ss'}(x,\ y) = -\,i\int d\lambda d\lambda' f_{ss'}(\lambda,\ \lambda')\,\varphi_\lambda(x)\,\varphi^*_{\lambda'}(y), \tag{2.7}$$

$$f_{ss'}(\lambda,\ \lambda') = \frac{1}{\langle S\rangle_0}\langle a^*_{s'}(\lambda')\,a_s(\lambda)S\rangle_0\,. \tag{2.8}$$

If the dynamical variable λ has a definite meaning even in the presence of interaction of the electrons with the phonons, then

$$f_{ss'}(\lambda,\ \lambda') = f_{ss'}(\lambda)\,\delta(\lambda-\lambda'), \tag{2.9}$$

$$G_{ss'}(x,\ y) = -\,i\int f_{ss'}(\lambda)\,\varphi_\lambda(x)\,\varphi^*_\lambda(y)\,d\lambda. \tag{2.10}$$

Passing to the limit $x_0\to y_0$ in accord with Eq. (2.5), we see that, to find the distribution of the electrons over the values of the "quantum number" λ, it suffices to expand $G_{ss'}(x,y)$ in terms of the eigenfunctions $\varphi_\lambda(x)\varphi^*_{\lambda'}(y)$; in such a case, it will immediately become clear whether or not it is possible to speak generally of such a distribution (i.e., whether the given dynamical variable makes sense). Similarly, for $x_0 > y_0$, the expansion of the Green's function in terms of the $\varphi_\lambda(x)\varphi^*_{\lambda'}(y)$ gives the distribution of "free places".

3. EQUATION FOR THE GREEN'S FUNCTION AND FOR THE "POTENTIAL"

The equations for the functions $G_{ss'}(x,y)$ and $a(x)$ can easily be obtained (if only in the model of Anderson[5]). With the use of Eq. (2.3), we have*

$$G_{s_1s_2}(x,\ y) = iK_{s_1s_2}(x,\ y) - g\int dzK_{s_1s'}(x,\ z)\left\{\frac{\delta G_{s's_2}(z,\ y)}{\delta\rho(z)} + ia(z)\,G_{s's_2}(z,\ y)\right\}, \tag{3.1}$$

$$a(x) = i\int dzF(x,\ z)\,\rho(z) \tag{3.2}$$

(Summation is carried out over the repeated spin indices). Here $K_{ss'}(x,y)$ and $F(x,y)$ are the distribution functions of the "free" electron and of the phonon field:

$$K_{ss'}(x,\ y) = \langle T\{\psi_s(x)\,\psi^*_{s'}(y)\}\rangle_0, \tag{3.3}$$

$$F(x,\ y) = \langle T\{\varphi(x)\,\varphi(y)\}\rangle_0 \tag{3.4}$$

Writing $\psi_s(x)$ in the form (2.6), and introducing the notation

$$f^{(0)}_{ss'}(\lambda) = \langle a^*_s(\lambda)\,a_{s'}(\lambda)\rangle_0, \tag{3.5}$$

we obtain

$$K_{ss'}(x,\ y) = \begin{cases} \int d\lambda\varphi_\lambda(x)\,\varphi^*_\lambda(y)\,\{\delta_{ss'} - f^{(0)}_{ss'}(\lambda)\}, & x_0 > y_0, \\ -\int d\lambda\varphi_\lambda(x)\,\varphi^*_\lambda(y)\,f^{(0)}_{ss'}(\lambda), & x_0 < y_0. \end{cases} \tag{3.6}$$

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* In Eq. (3.2), the components corresponding to closed loops are omitted. It is easy to show, however, that in the approxomation employed below, consideration of such terms leads only to the renormalization of the Fermi energy. Assuming the latter operation to be completed, one can omit this term.

[5] J. L. Anderson, Phys. Rev. **94**, 703 (1954).

This expression can be written in more compact form if the functions $\varphi_\lambda(x)$ describe stationary states (this particular case also represents fundamental interest). Let $W(\lambda)$ be the energy eigenvalue which corresponds to the "quantum number" λ. Then $\varphi_\lambda(x) = U_\lambda(\mathrm{x}) \exp\{-iW(\lambda)x_0\}$ and, as can easily be seen, the function $K_{ss'}(x, y)$ can be written in the form

$$K_{ss'}(x, y) = \frac{i}{2\pi} \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int d\lambda U_\lambda(\mathrm{x})\, U_\lambda^*(\mathrm{y}) \tag{3.7}$$

$$\times \int_{-\infty}^{+\infty} ds e^{-is(x_0-y_0)} \frac{\delta_{ss'}[s-W(\lambda)] + i\varepsilon[2f^0_{ss'}(\lambda) - \delta_{ss'}]}{[s-W(\lambda)]^2+\varepsilon^2}.$$

This formula is suitable for use if the structure function $f^{(0)}_{ss'}(\lambda)$ is continuous. In some cases, however (for example, in the investigation of a degenerate electron gas), this function is discontinuous:

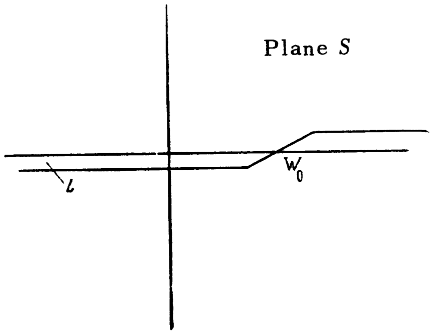
$$f^{(0)}_{ss'}(\lambda) = \begin{cases} \delta_{ss'}, & W < W_0, \\ 0, & W > W_0, \end{cases} \tag{3.8}$$

where W_0 is some limiting value. It is then appropriate to replace Eq. (3.7) with

$$\times K_{ss'}(x, y) = \frac{i}{2\pi}\delta_{ss'} \tag{3.9}$$

$$\int d\lambda U_\lambda(\mathrm{x})\, U_\lambda(\mathrm{y}) \int_L ds \frac{\exp\{-is(x_0-y_0)\}}{s-W(\lambda)},$$

where the contour of integration L is drawn in the figure. (This formula is entirely analogous to the well known expression for the function $S_c(x-y)$[6].)



The quantum function of the phonon field $A(x)$ has the form[3]

$$A(x) = \frac{1}{\sqrt{2\pi^3}} \int \frac{d\mathrm{f}}{|\mathrm{f}|} \{b_\mathrm{f} e^{ifx} + b_\mathrm{f}^* e^{-ifx}\}. \tag{3.10}$$

Here b_f, b_f^* are the usual Bose operators, $fx = \mathrm{f}\cdot\mathbf{x} - |\mathrm{f}|x_0$ and the possible values of the quasi-momentum of the phonon, $|\mathrm{f}|$, are bounded above by the Debye value f_0. Introducing the expression

$$B(\mathrm{f}) = \langle b_\mathrm{f}^* b_\mathrm{f}\rangle_0, \tag{3.11}$$

we obtain

$$F(x, y) = \begin{cases} \frac{1}{2\pi^3} \int d\mathrm{f}\,|\mathrm{f}|\,\{2iB(\mathrm{f})\sin(f, x-y) + e^{i(f, x-y)}\}, & x_0 > y_0, \\ \frac{1}{2\pi^3} \int d\mathrm{f}\,|\mathrm{f}|\,\{-2iB(\mathrm{f})\sin(f, x-y) + e^{-i(f, x-y)}\}, & x_0 < y_0. \end{cases} \tag{3.12}$$

In what follows, it will be more appropriate to use these integral representations for the functions $K_{ss'}(x, y)$ and $F(x, y)$ and not the explicit expressions for them.

4. "OPERATOR OF THE ENERGY OF INTERACTION." THEORY OF WEAK COUPLING

For the application of any particular form of perturbation theory, it is convenient to deal with the functional derivative not of the Green's function itself but of its inverse matrix. It is easy to see that

$$\delta G_{s's_2}(z, y)/\delta\rho(z)$$

$$= -\int dz'dx'dx'' \frac{\delta a(z')}{\delta\rho(z)} G_{s's'''}(z, x'') \tag{4.1}$$

$$\times \frac{\delta G^{-1}_{s'''s''}(x'', x')}{\delta a(z')} G_{s''s_2}(x', y).$$

If we introduce the notation

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[6] A. I. Akhiezer and B. V. Berestetskii, *Quantum Electrodynamics*, Moscow, 1953.

$$\Delta E_{s's''}(z,\ x') \tag{4.2}$$
$$=ig\int dz'dx''\,\frac{\delta a(z')}{\delta\rho(z)}\,G_{s's'''}(z,\ x'')\,\frac{\delta G^{-1}_{s'''s''}(x'',\ x')}{\delta a(z')},$$

then Eq. (3.1) reduces to the form

$$G_{s_1s_2}(x,\ y)=iK_{s_1s_2}(x,\ y)-ig \tag{4.3}$$
$$\times\int dzK_{s_1s'}(x,\ z)\,G_{s's_2}(z,\ y)\,a(z)$$
$$-\,i\int dzdx'K_{s_1s'}(x,\ z)\,\Delta E_{s's''}(z,\ x')\,G_{s''s_2}(x',\ y).$$

Multiplying Eq. (4.3) by G^{-1} and varying the result with respect to $a(z')$, we get the expression

$$\int K_{s''s'''}(z,\ x'')\,\frac{\delta G^{-1}_{s'''s''}(x'',\ x')}{\delta a(z')}\,dx'' \tag{4.4}$$
$$=gK_{s's''}(z,\ x')\,\delta(x'-z')$$
$$+\int dz''K_{s's'''}(z,\ z'')\,\frac{\delta\Delta E_{s'''s''}(z'',\ x')}{\delta a(z')}.$$

which is of use in what follows. The matrix $\Delta E_{ss'}(z,\ x')$ characterizes the additional energy which is transformed by the electron into the phonon field and itself represents the analogue of the mass operator of quantum field theory. In this connection it is pertinent to call ΔE the operator of the energy of interaction.

To solve the nonlinear equation (4.3), we must make use of some form of perturbation theory. In the present work, we limit ourselves to the case of weak coupling. It should be borne in mind, however, that even under these conditions, direct expansion of the Green's function in a power series of the coupling constant is, generally speaking, hazardous. It is more accurate to expand not the Green's function itself, but the equation by which it is defined, i.e., the operator of the energy of interaction ΔE (or the inverse matrix G^{-1}, which amounts to the same thing).

In accordance with what we have said, we write

$$\Delta E=\Delta E^{(0)}+g\Delta E^{(1)}+g^2\Delta E^{(2)}+\cdots \tag{4.5}$$

Noting that, by Eq. (3.2),

$$\frac{\delta a(x)}{\delta\rho(z)}=iF(x,z);\quad a(x)\,|_{\rho=0}=0, \tag{4.6}$$

we get (the indices on G denote that this function satisfies the equation expanded in corresponding fashion):

$$\Delta E^{(0)}=\Delta E^{(1)}=0, \tag{4.7}$$

$$G^{(0)}_{s_1s_2}(x,\ y)=G^{(1)}_{s_1s_2}(x,\ y)=iK_{s_1s_2}(x,\ y), \tag{4.8}$$

$$\Delta E^{(2)}_{s's''}(z,\ x')$$
$$=-\frac{i}{g}\int dz'dx''F(z',\ z)\,K_{s's'''}(z,\ x'')\,\frac{\delta G^{-1}_{s'''s''}(x'',\ x')}{\delta a(z')}_{g\to 0}. \tag{4.9}$$

By reason of Eq. (4.4), Eq. (4.9) takes the form

$$\Delta E^{(2)}_{s's''}(z,\ x')=-iF(x',\ z)\,K_{s's''}(z,\ x'), \tag{4.10}$$

and correspondingly, we find the following linear equation for $G^{(2)}_{s_1s_2}(x,\ y)$:

$$G^{(2)}_{s_1s_2}(x,\ y)$$
$$=iK_{s_1s_2}(x,\ y)-g^2\int dz'dzK_{s_1s'}(x,\ z)\,F(z',\ z)\,K_{s's''}(z,\ z')\,G^{(2)}_{s''s_2}(z',\ y). \tag{4.11}$$

This equation can be solved exactly.

5. FERMI DISTRIBUTION FOR AN ELECTRON GAS IN A CRYSTAL AT ABSOLUTE ZERO

The general relations obtained above permit us to solve without difficulty the problem of the effect of the zero vibrations of the crystalline lattice on the electron distribution function. As a first step, we "smooth out" the periodic potential of the lattice, considering it only formally – by replacing the real mass of the electron by an effective mass (m). For the functions $\varphi_\lambda(x)$ it is reasonable to take plane waves (the index λ denotes the three components of the momentum of the electron):

$$\varphi_\lambda(x) = (2\pi)^{-3/2} e^{i\mathbf{p}\mathbf{x} - ip_0x_0}; \tag{5.1}$$

$$p_0 \equiv W(\lambda) = \mathbf{p}^2/2m.$$

The structure function $f^{(0)}_{ss'}(\lambda)$ has the form of Eq. (3.8), where $W_0 = E_F = p_F^2/2m$ is the Fermi energy. The function $B(f)$ is zero at absolute zero. Thus the distribution functions $K_{ss'}(x, y)$ and $F(x, y)$ take the form

$$K_{ss'}(x, y) = \delta_{ss'}K(x-y), \tag{5.2}$$

$$K(x-y) = \frac{i}{(2\pi)^4}\int d\mathbf{p}\int_L ds\, \frac{\exp\{i(\mathbf{p}, \mathbf{x}-\mathbf{y}) - is(x_0-y_0)\}}{s - p^2/2m}, \tag{5.3}$$

$$F(x-y) = \frac{1}{2\pi^3}\int d\mathbf{f}\,|\mathbf{f}|\,\exp\{i(\mathbf{f}, \mathbf{x}-\mathbf{y}) - i|\mathbf{f}||x_0-y_0|\}.. \tag{5.4}$$

Equation (4.11) is easily solved in this case. First, it is clear that the Green's function is diagonal in the spin indices and depends only on the difference of the arguments

$$G^{(2)}_{s_1s_2}(x, y) = \delta_{s_1s_2}G(x-y). \tag{5.5}$$

From Eq. (4.11) we get for the function $G(x-y)$

$$G(x-y) = iK(x-y) - g^2 \times \int dz\,dz'\,K(x-z)F(z'-z)K(z-z')G(z'-y). \tag{5.6}$$

The boundary condition in this equation is clear from a consideration of Eq. (5.3). Setting $s = E_F + s'$ in that case, we see that in the new variables $(\mathbf{p}, s')$ the boundary condition takes the usual form: for $x_0 > y_0$ we must get diverging waves, for $x_0 < y_0$, converging (one does not have to pay attention to the factor $\exp\{-iE_F(x_0-y_0)\}$, which appears in front of the integral, in this case).

The solution of Eq. (5.6) for the given boundary condition has the form

$$G(x-y) = -\frac{1}{(2\pi)^4} \int d\mathbf{p}\int_L dp_0\, \frac{\exp\{i(\mathbf{p}, \mathbf{x}-\mathbf{y}) - ip_0(x_0-y_0)\}}{p_0 - (p^2/2m) + ig^2f(\mathbf{p}, p_0)}, \tag{5.7}$$

where the contour L is drawn in the figure (for $W_0 = E_F$), and

$$f(\mathbf{p}, p_0) = \int K(x)F(-x)\exp\{-i\mathbf{p}\mathbf{x} + ip_0x_0\}\,dx. \tag{5.8}$$

Calculation of the integral gives (we limit ourselves to the case $p_F > f_0$):

1) for $p_F - f_0 < p < p_F + f_0$:

$$\begin{aligned} f(\mathbf{p}, p_0) = -\frac{im}{\pi^2 p}\Bigg\{ & \frac{f_0^3}{3}\ln\frac{(f_0+p)^2 + 2m(p_0+f_0)}{(f_0-p)^2 + 2m(p_0-f_0)} + \frac{f_0^3}{3}\ln\frac{E_F + p_0 - f_0}{E_F + p_0 + f_0} \\ & + (p+m)\left[\frac{(p+m)^2}{3} + |a|^2\right]\ln\frac{(f_0+p)^2 + 2m(p_0+f_0)}{p_F^2 + 2m(p_0 + p_F - p)}\,\frac{(f_0-p)^2 + 2m(p_0-f_0)}{p^2 + 2mp_0} \\ & + (p-m)\left[\frac{(p-m)^2}{3} + |b|^2\right]\ln\frac{p_F^2 + 2m(p_0 - p_F + p)}{p^2 + 2mp_0} \\ & + a\left[\frac{|a|^2}{3} + (p+m)^2\right]\ln\left(\frac{f_0 + p + m + a}{f_0 + p + m - a}\,\frac{p_F + m - a}{p_F + m + a}\,\frac{p + m - a}{p + m + a} \right. \\ & \left. \times\frac{p + m - f_0 + a}{p + m - f_0 - a}\right) + b\left[\frac{|b|^2}{3} + (p-m)^2\right]\ln\left(\frac{p_F - m + b}{p_F - m - b}\,\frac{p - m - b}{p - m + b}\right) \\ & + \frac{(E_F + p_0)^3}{3}\ln\frac{(E_F + p_0)^2 - (p_F - p)^2}{(E_F + p_0)^2 - f_0^2} + \frac{8}{3}mp(p_F - p) \\ & + 2p(p_F - p)^2 + [f_0^2 - (p_F - p)^2]\left[2(p+m) - \frac{E_F + p_0}{3}\right] \\ & - \frac{4}{3}[p^2m + (p+m)f_0^2 - mp_F^2]\Bigg\}, \end{aligned} \tag{5.9}$$

where

$$a = \sqrt{m^2 - 2m(p_0 - p)};\quad b = \sqrt{m^2 - 2m(p_0 + p)};\quad p = |\mathbf{p}|.$$

2) for $p < p_F - f_0$:

$$\begin{aligned} f(\mathbf{p}, p_0) = & -\frac{im}{p\pi^2}\left\{\frac{f_0^3}{3}\ln\frac{(p+f_0)^2 + 2m(p_0 - f_0)}{(p-f_0)^2 + 2m(p_0 - f_0)}\right. \\ & + (p-m)\left[\frac{(p-m)^2}{3} + |b|^2\right]\ln\frac{(p+f_0)^2 + 2m(p_0 - f_0)}{p^2 + 2mp_0} \\ & + (p+m)\left[\frac{(p+m)^2}{3} + |a|^2\right]\ln\frac{(p-f_0)^2 + 2m(p_0 - f_0)}{p^2 + 2mp_0} \\ & + b\left[(p-m)^2 + \frac{|b|^2}{3}\right]\ln\frac{p+f_0-m+b}{p+f_0-m-b}\,\frac{p-m-b}{p-m+b} \\ & \left. - a\left[(p+m)^2 + \frac{|a|^2}{3}\right]\ln\frac{f_0+a-p-m}{f_0-a-p-m}\cdot\frac{p+m+a}{p+m-a} + \frac{16}{3}mpf_0 + \frac{2}{3}pf_0^2\right\}. \end{aligned} \tag{5.10}$$

3) for $|\ p > p_\Phi + f_0$:

$$\begin{aligned} f(\mathbf{p}, p_0) = & -\frac{im}{p\pi^2}\left\{\frac{f_0^3}{3}\ln\frac{(f_0+p)^2 + 2m(p_0 + f_0)}{(f_0-p)^2 + 2m(p_0 + f_0)}\right. \\ & + (p-m)\left[\frac{(p-m)^2}{3} + |b|^2\right]\ln\frac{(p-f_0)^2 + 2m(p_0 + f_0)}{p^2 + 2mp_0} \\ & + (p+m)\left[\frac{(p+m)^2}{3} + |a|^2\right]\ln\frac{(p+f_0)^2 + 2m(p_0 + f_0)}{p^2 + 2mp_0} \\ & + b\left[(p-m)^2 + \frac{|b|^2}{3}\right]\ln\frac{f_0-p+m-b}{f_0-p+m+b}\,\frac{p-m-b}{p-m+b} \\ & \left. - a\left[(p+m)^2 + \frac{|a|^2}{3}\right]\ln\frac{f_0+p+m-a}{f_0+p+m+a}\,\frac{p+m+a}{p+m-a} + \frac{2}{3}pf_0^2 - \frac{16}{3}mpf_0\right\}. \end{aligned} \tag{5.11}$$

Comparing Eq. (5.7) with Eq. (2.10), we can find the electron momentum distribution function $f(\mathbf{p})$ where the interaction with the vibrations of the lattice is taken into account. It will be more useful for us, however, to compute not $f(\mathbf{p})$ but the hole distribution function $\Phi(\mathbf{p})$ which is related to $f(\mathbf{p})$ by the relation $\Phi(\mathbf{p}) = 1 - f(\mathbf{p})$. From Eq. (5.7) we get

$$\Phi(\mathbf{p}) = \frac{i}{2\pi}\int_L dp_0\,\frac{\exp\{-ip_0(x_0 - y_0)\}}{p_0 - (\mathbf{p}^2/2m) + ig^2 f(\mathbf{p}, p_0)} \tag{5.12}$$

(for $x_0 > y_0$; $x_0 \to y_0$).

Comparing Eq. (5.7) with Eqs. (4.10) and (5.8), it is easy to see that the pole of the integrand in Eq. (5.7) determines the energy $W(\mathbf{p})$ of the electron which interacts with the phonons:

$$W(\mathbf{p}) = (\mathbf{p}^2/2m) - ig^2 f(\mathbf{p}, \mathbf{p}_0^*), \tag{5.13}$$

where p_0^* is the square root of the denominator in Eq. (5.7). We note that the following inequalities are practically always valid:

$$E_F \gg p_F;\quad f_0 \gg m. \tag{5.14}$$

Therefore the expression for $W(\mathbf{p})$ can be written in the comparitively simple form:

$$\begin{aligned} W(p) = & \frac{p^2}{2m} - \frac{g^2}{\pi^2}\,\frac{m}{p}\left\{\frac{2}{3}pf_0^2 + \frac{f_0^3}{3}\ln\frac{f_0^2 + 2pf_0 + 2p^2}{f_0^2 - 2pf_0 + 2p^2}\right. \\ & \left. + \frac{4}{3}p^3\ln\frac{f_0^2 + 4p^2}{4p^2} + \frac{8}{3}p^3\left[\operatorname{arc\,tg}\frac{p+f_0}{p} + \operatorname{arc\,tg}\frac{p-f_0}{p} - \frac{\pi}{2}\right]\right\}. \end{aligned} \tag{5.15}$$

(This formula is valid for $p \gg m$.)

It is further clear from Eq. (5.14) that for

$p < E_F + {}^1\!/_2\, m$ all the branch points of $F(\mathrm{p}, \mathrm{p}_0)$ (considered as a function of the complex variable p_0) lie to the left of E_F. By drawing the curve along the real axis from $-\infty$ to $p + 1/2\, m$, we can compute the integral (5.12) directly for values of p in the interval $0 \leqslant p < E_F + {}^1\!/_2\, m$ (i.e., in all regions of practical interest). It must only be kept in mind that the pole of the integrand in Eq. (5.12) is generally simple (because reduction to zero (for one and the same value of p_0) both of the denominator itself and its derivative with respect to p_0 is possible only for a certain definite g). Thus we obtain for $p < E_F + {}^1\!/_2\, m$ (here and below, we keep only the first terms of the expansion in small quantities of the type m/p_F, etc.):

$$\Phi(\mathrm{p}) = 0, \quad p < p_c; \tag{5.16}$$

$$\Phi(\mathrm{p}) = \left\{1 + \frac{2g^2m^2}{\pi^2}\left[\ln\frac{f_0^4 + 4p^4}{4p^4} + 2\left(\operatorname{arc\,tg}\frac{p + f_0}{p} + \operatorname{arc\,tg}\frac{p - f_0}{p} - \frac{\pi}{2}\right) + \frac{4}{3}\,\frac{4p^2f_0^2 - f_0^4}{f_0^4 + 4p^4}\right]\right\}^{-1}, \quad p > p_c \tag{5.17}$$

where

$$p_c = p_F\left\{1 + \frac{2g^2m^2}{\pi^2}\left[\frac{2}{3}t^2 + \frac{t^3}{3}\ln\frac{t + 2 + 2t^2}{t - 2 + 2t^2} + \frac{4}{3}\ln\left(1 + \frac{t^4}{4}\right) + \frac{8}{3}\left(\operatorname{arc\,tg}(1 + t) + \operatorname{arc\,tg}(1 - t) - \frac{\pi}{2}\right)\right]\right\}, \tag{5.18}$$

$$t = f_0/p_F.$$

In particular, if $p_F \gg f_0$ (generally speaking this inequality does not occur), then Eq. (5.17) has the simple form

$$\Phi(\mathrm{p}) = \left[1 + \frac{2g^2m^2}{3\pi^2}\,\frac{f_0^2}{p^2}\right]^{-1} \tag{5.19}$$

(if $p > p_c$). In this case,

$$p_c = p_F\left\{1 - \frac{4}{3}\,\frac{g^2m^2}{\pi^2}\,t^2\right\}. \tag{5.20}$$

It follows from Eq. (5.17) that, because of the interaction with the zero vibrations of the lattice, the electron momentum distribution is somewhat "smeared out"; even at absolute zero some of the electrons possess momentum which exceeds the Fermi limit. Analogous results are obtained also in the calculation of the interaction of electrons with optical vibrations.

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