Application of the Theory of Stochastic Processes to the Investigation of Nuclear Fission

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The statistical theory of the ionization chamber is considered for the investigation of nuclear fission. The mean number of pulses produced by the ∝ particle background is determined. A theory of random processes is developed in connection with the investigation of this stochastic process.

In the construction of ionization chambers for the study of fission processes, the problem of excluding (or at least minimizing) the ionizing action of ∝-particles occupies a central place. The energy of the fission fragments can vary within wide limits, but their mean energy, in each case, is an order of magnitude larger than the energy of the ∝-particles. However, the superposition of pulses from several ∝-particles can create a pulse which is comparable with those from the fragments. The pulses are recorded as a pulse which is created by the fission fragments; therefore, it is necessary to give at least an approximate estimate of the number of such cases.

The problem of the correction of the number of fissions was considered by Rossi and Staub.[1] Under the assumption that the pulses have rectangular shape and duration τ, they determined the number of readings per unit time which contained unwanted pulses of ∝-particles. For practical purposes, their formula is satisfied only approximately, since in most cases the pulses have an exponential shape. The purpose of the present research was to consider the effect of the exponential shape on the number of readings.

1. Let $H(x)$ be the probability that the amplitude of the pulse produced by the ∝-particles be less than x. For simplicity, we assume that the number of pulses satisfies a Poisson distribution.* If N is the pulse density and $R(x,t)$ is the probability that the sum of pulses produced by the ∝-particles in the time interval $(0,t)$ is less than x at the time t, one can easily convince oneself that $R(x,t+\Delta t)$ is composed of the probabilities of two mutually incompatible events, as follows:

1) either the sum of pulses is less than $xe^{\Delta t/\tau}$ at time t, and no new pulse appears in the time interval $(t,t+\Delta t)$,

2) or, the sum of pulses is less than $(x-y)e^{\Delta t/\tau}$ at the time t and a new pulse appears in the time interval $(t,t+\Delta t)$ with amplitude lying in the interval $(y,y+dy)$. It is evident** that

$$R(x,t+\Delta t)=(1-N\Delta t)R(xe^{\Delta t/\tau},t) \quad (1)$$
$$+N\Delta t\int_0^x R[(x-y)e^{\Delta t/\tau},t]\,dH(y)+o(\Delta t).$$

It follows from Eq. (1) that

$$\frac{\partial R}{\partial t}=\frac{x}{\tau}\frac{\partial R}{\partial x} \quad (2)$$
$$+N\left\{\int_0^x R(x-y,t)\,dH(y)-R(x,t)\right\}$$

If $\lim R(x,t)\to R(x)$ as $t\to\infty$, then the following relation results from Eq. (2):

$$\frac{x}{\tau}\frac{dR}{dx}=N\left\{R(x)-\int_0^x R(x-y)\,dH(y)\right\} \quad (3)$$

By virtue of the independence of the pulses of ∝-particles from the pulses of fission fragments, the distribution function of the sum of pulses of fission fragments $S(x)$ is defined in similar fashion by the equation

$$\frac{x}{\tau}\frac{dS}{dx}=M\left\{S(x)-\int_0^x S(x-y)\,dK(y)\right\}, \quad (4)$$

where $K(y)$ is the pulse amplitude distribution function and M is the pulse density. Knowing the distribution functions $R(x)$ and $S(x)$, we can determine the distribution function $P(x)$ of the output voltage of the ionization chamber:

$$P(x)=\int_0^x R(x-y)\,dS(y). \quad (5)$$

*It is shown in Appendix A that these results are easily generalized to the case of certain non-Markovian processes. A similar generalization has been given by Takacs.[2]

[1] B. B. Rossi and H. H. Staub, *Ionization Chambers and Counters*, New York, 1949.

[2] L. Takacs, MTA III, Oszt. Közlemenyei. 4, 473 (1954).

**The probability that more than one pulse appears in the time interval $(t, t+\Delta t$ is equal to $o(\Delta t)$.

We denote by V the recording level of the discriminator which follows the ionization chamber. If the voltage at the input of the discriminator increases above the recording level V, this fact is recorded as a "nuclear fission". Naturally, in this case we also count those pulses which are connected with the superposition of pulses from α-particles. We denote by Q_1 the density of pure fissions and by Q_2 the density of pulses connected with the superposition of pulses from α-particles. It is evident that the number of readings Q is equal to $Q_1 + Q_2$. The quantities Q_1 and Q_2 are determined by the following expressions:

$$Q_1 = M \int_0^V \{1 - K(y)\} dP(V-y); \quad (6)$$

$$Q_2 = N \int_0^V \{1 - H(y)\} dP(V-y). \quad (7)$$

By appropriate choice of the recording level V and other parameters, one can also bring about the condition that $Q_1 \gg Q_2$.

2. Let us choose the amplitude distribution function of the α-particles in the following form:

$$H(x) = 1 - e^{-ax}, \quad (8)$$

where $1/a$ is the mean value of the amplitude of the pulses. Since the ionizing effects of the fission fragments are cancelled out, we obtain the following expression for the distribution function $K(x)$:

$$K(x) = b \int_0^x \{1 - e^{-b(x-y)}\} e^{-by} dy, \quad (9)$$

where $2b^{-1}$ is the mean value of the amplitude of the pulses of fission fragments. Making use of Eq. (8), we get from Eq. (3):

$$d \ln \bar{r}(z)/dz = - N\tau (z+a)^{-1}, \quad (10)$$

where $$r(z) = \int_0^\infty e^{-zx} dR(x).$$

We then obtain immediately

$$\bar{r}(z) = a^{N\tau} (z+a)^{-N\tau}. \quad (11)$$

The original function $r(x)$ is defined in elementary fashion:

$$r(x) = a e^{-ax} (ax)^{N\tau - 1} / \Gamma(N\tau), \quad (12)$$

and the distribution function $R(x)$ has the form:

$$R(x) = \Gamma(ax, N\tau) / \Gamma(N\tau), \quad (13)$$

where $\Gamma(ax, N\tau)$ is the so called incomplete gamma function.

The function $s(x)$ is defined in similar fashion. It is easy to show that $s(z)$ satisfies the following equation:

$$d \ln \bar{s}(z)/dz = - M\tau \{(z+b)^{-1} + b(z+b)^{-2}\}. \quad (14)$$

From Eq. (14) we obtain

$$\bar{s}(z) = b^{M\tau} \times \exp\{- M\tau z (z+b)^{-1}\} (z+b)^{-M\tau}, \quad (15)$$

whence

$$s(x) = b(xb/M\tau)^{(M\tau-1)/2} \times \exp\{-(M\tau + bx)\} I_{M\tau-1}(2\sqrt{M\tau b x}), \quad (16)$$

where

$$I_\nu(x) = i^{-\nu} J_\nu(ix).$$

The basic problem consists in the determination of the function $Q(V)$. Making use of the results of Eq. (5) and Eq. (11), we find from Eqs. (6) and (7) that

$$\bar{Q}_1(z) = (M/N) \bar{Q}_2(z)(z+a) \times (z+b)^{-1} \{1 + b(z+b)^{-1}\}; \quad (17)$$

$$\bar{Q}_2(z) = N a^{N\tau} b^{M\tau} \exp\{- M\tau z (z+a)^{-1}\} \times (z+a)^{-(N\tau+1)} (z+b)^{-M\tau}, \quad (18)$$

where

$$\bar{Q}_1(z) = \int_0^\infty e^{-zx} Q_1(x) dx;$$

$$\bar{Q}_2(z) = \int_0^\infty e^{-zx} Q_2(x) dx.$$

To simplify the formula, we introduce the notation $N\tau = \alpha$; $M\tau = \beta$. It should be noted that $\alpha > \beta$ and $\alpha \ll 1$. To determine $Q_1(V)$ and $Q_2(V)$ we must calculate the following integrals (see appendix B):

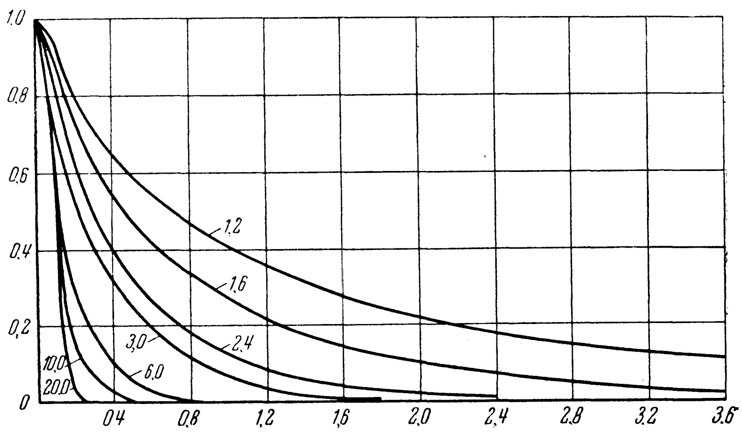

FIG. 1. Dependence of the ratio $MQ_2(N)/NQ_1(V)$ on the potential V of the discriminator (horizontal axis) for different values of a which are shown on the curves). The potential is in units of b^{-1}

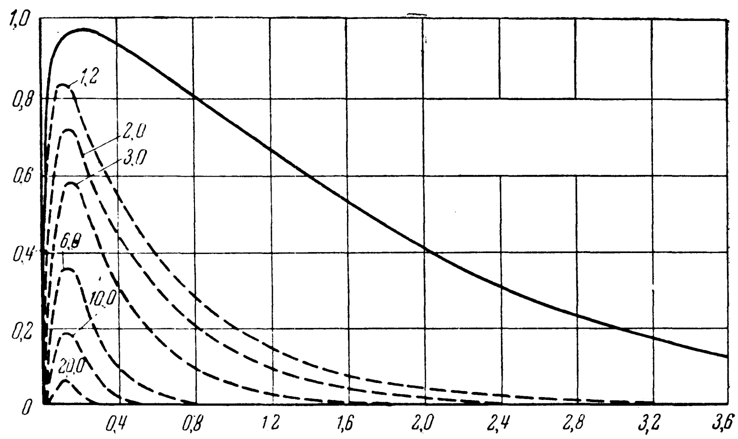

FIG. 2. The solid curves show the dependence of the number of true fissions (Q_1M^{-1}), and the broken lines. The dependence of the number of false fissions, associated with the superposition of α-particles pulses (Q_2N^{-1}) on the potential V of the discriminator (horizontal axis) for different values of a, shown on the curves, in the case $M\tau = 10^{-4}$ $N\tau = 10^{-3}$. The potential is in units of b^{-1}

$$Q_2(x) = \frac{Na^{\alpha}b^{\beta}e^{-\beta}}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \exp\{zx+\beta b(z+b)^{-1}\}(z+b)^{-\beta}(z+a)^{-(\alpha+1)}dz, \quad (19)$$

and

$$Q_1(x) = u_{11}(x) + u_{12}(x),$$

$$u_{11}(x) = (2\pi i)^1 \frac{M}{N}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{zx}Q_2(z)(z+a)(z+b)^1 dz; \quad (20)$$

$$u_{12}(x) = (2\pi i)^1 \frac{M}{N}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{zx}Q_2(z)(z+a)(z+b)^2 dz. \quad (21)$$

By virtue of the fact $V\beta b \ll 1$ for values of V of practical importance, we can obtain approximate expressions for $Q_1(V)$ and $Q_2(V)$ in the form

$$Q_1(V) \sim M \exp\{-(M\tau + bV)\} \times (Va)^{N\tau}(Vb)^{M\tau}(1 + bV); \tag{22}$$

$$Q_2(V) \sim N \exp\{-(M\tau + aV)\} \times (Va)^{N\tau}(Vb)^{M\tau}. \tag{23}$$

Equation (22) gives the number of readings of true fissions, and Eq. (23) gives the number of false fissions, connected with the superposition of pulses from α-particles, both per unit time.

In Fig. 1 we have plotted the dependence of $MQ_2(V)/NQ_1(V)$ on the voltage of the discriminator for a given value of M/N and for difference values of a/b. It should be noted that the voltage b^{-1} was chosen as the unit voltage. If the value of a/b is known from experimental data, then we can easily find the best recording conditions with the aid of Fig. 1. The curves in Fig. 2 show the dependence of the number of true fissions and false fissions (dotted curves) connected with the superposition of pulses from α-particles, on the voltage of the discriminator for various values of a/b (b^{-1} is taken here to be of unit voltage).

It should be noted that these calculations have a much more general character and can be used successfully in the analysis of the results of an arbitrary discriminator amplitude.

At the present time there are no reliable experimental data on the distribution law of the amplitudes of pulses from α-particles and fission fragments, and therefore we used an exponential (as a very general) distribution law.

APPENDIX A

Let $F(t)$ be the distribution function of time intervals between two successful voltage pulses. If we note by $G_n(x)$ the probability that the voltage on the output resistor at the time of time of realization of the nth pulse be less than x, and consider an exponential decay (with the law $e^{-t/\tau}$) of the voltage between two successive pulses, then we can describe $G_n(x)$ by the following recurrence relation:

$$G_n(x) = \int_0^\infty \int_0^x G_{n-1}[(x-y)e^{t/\tau}]\, dF(t)\, dH(y). \tag{A1}$$

With the help of the Laplace-Stieltjes representation, we get from Eq. (A1):

$$\bar{g}_n(z) = \bar{h}(z) \int_0^\infty \bar{g}_{n-1}(ze^{-t/\tau})\, dF(t); \tag{A2}$$

$$\bar{g}_n(z) = \int_0^\infty e^{-zx}\, dG_n(x), \tag{A3}$$

$$\bar{h}(z) = \int_0^\infty e^{-zx}\, dH(x).$$

If $\lim \bar{g}_n(z) \to \bar{g}(z)$ for $n \to \infty$, then we have the following integral equation for $\bar{g}(z)$:

$$\bar{g}(z) = \bar{h}(z) \int_0^\infty \bar{g}(ze^{-t/\tau})\, dF(t). \tag{A4}$$

From Eq. (A4) we can determine the distribution function $G(x)$, with the help of which we can easily find the distribution function $R(x)$, i.e.,

$$R(x) = \lambda \int_0^\infty \{1 - F(t)\}\, G(xe^{t/\tau})\, dt, \tag{A5}$$

where

$$\lambda = \int_0^\infty \{1 - F(t)\}\, dt. \tag{A6}$$

For an exponential distribution of the intervals between successive pulses ($F(t) = 1 - e^{-Nt}$) Eq. (A6) coincides with Eq. (3) from Sec. 1.

APPENDIX B

In Eqs. (19)-(21) it was necessary to determine integrals of the following type:

$$\Phi(x) = (2\pi i)^{-1} \times \int_{\sigma - i\infty}^{\sigma + i\infty} \exp\{zx + \beta b(z+b)^{-1}\} (z+a)^{-(r+\alpha)}(z+b)^{-(l+\beta)}\, dz. \tag{B1}$$

Assuming that $\beta bx \ll 1$ and $\beta < \alpha \ll 1$, and with the aid of the rapidly converging expansion

$$\exp\{\beta b(z+b)^{-1}\} = \sum_{m=0}^{\infty} \frac{(\beta b)^m}{m!}(z+b)^{-m}$$

calculation of the integral (B1) reduces to the determination of the following integral:

$$\Phi_m(x) = (2\pi i)^{-1} \times \int_{\sigma - i\infty}^{\sigma + i\infty} e^{zx}(z+a)^{-(r+a)}(z+b)^{-(m+l+\beta)}\, dz. \tag{B2}$$

It is not difficult to show that the integral (B2) has the form

$$\Phi_m(x) = e^{-ax} \frac{x^{\nu_m+\mu-1}}{\Gamma(\nu_m+\mu)} {}_1F_1(\nu_m, \nu_m+\mu, cx), \tag{B3}$$

where

$$\nu_m = m + l + \beta, \quad \mu = r + \alpha, \quad \times\; c = a - b, \tag{B4}$$

and ${}_1F_1(\nu_m, \nu_m+\mu, cx)$ is the degenerate hypergeometric function (see Ref. 3). In this way, Eq. (B1) is represented in the form of a rapidly converging series:

$$\Phi(x) = e^{-ax} \times \sum_{m=0}^{\infty} \frac{(\beta b)^m}{m!} \frac{x^{\nu_m+\mu-1}}{\Gamma(\nu_m+\mu)} {}_1F_1(\nu_m, \nu_m+\mu, cx; \tag{B5}$$

Making use of Eq. (B5), we get for $Q_1(V)$ and $Q_2(V)$ the expressions

$$Q_1(V) = Ma^\alpha b^\beta \exp\{-(aV+\beta)\}\left\{\sum_{k=0}^{\infty} \frac{(\beta b)^k}{k!} \frac{V^{\alpha+\beta+k}}{\Gamma(\alpha+\beta+k+1)} \times {}_1F_1(\beta+k+1, \alpha+\beta+k+1;\ cV) + bV \sum_{k=0}^{\infty} \frac{(\beta V)^k}{k!} \frac{V^{\alpha+\beta+k}}{\Gamma(\alpha+\beta+k+2)} \times {}_1F_1(\beta+k+2, \alpha+\beta+k+2;\ cV)\right\} \tag{B6}$$

and

$$Q_2(V) = Na^\alpha b^\beta \exp\{-(aV+\beta)\} \sum_{k=0}^{\infty} \frac{(\beta b)^k}{k!} \frac{V^{\alpha+\beta+k}}{\Gamma(\alpha+\beta+k+1)} \times {}_1F_1(\beta+k, \alpha+\beta+k+1;\ cV). \tag{B7}$$

It is not difficult to be convinced that for sufficiently small values of α and β:

$${}_1F_1(\beta+k, \alpha+\beta+k;\ cV) \sim e^{cV} - \alpha \sum_{m=1}^{\infty} \frac{\lambda_m^k}{1+(\alpha+\beta)\lambda_m^k} \frac{(cV)^m}{m!}, \tag{B8}$$

where

$$\lambda_m^k = \sum_{j=k}^{k+m-1} \frac{1}{j}$$

Furthermore,

$${}_1F_1(\beta+k, \alpha+\beta+k+1; cV) \sim k\int_0^1 t^{k-1}e^{cVt}\,dt - \alpha k \sum_{m=1}^{\infty} \frac{\lambda_{m+1}^k}{(m+k)[1+(\alpha+\beta)\lambda_{m+1}^k]} \frac{(cV)^m}{m!} - \beta k \sum_{m=1}^{\infty} \frac{1}{(m+k)^2[1+(\alpha+\beta)\lambda_{m+1}^k]} \frac{(cV)^m}{m!}. \tag{B9}$$

If we neglect terms which are linear in α and β we get

$$Q_1(V) \sim M\exp\{-(M\tau+bV)\} \times (Va)^{N\tau}(Vb)^{M\tau}(1+bV), \tag{B10}$$

$$Q_2(V) \sim N\exp\{-(M\tau+aV)\} \times (Va)^{N\tau}(Vb)^{M\tau}. \tag{B11}$$

With the aid of Eqs. (B8) and (B9) we can estimate the inaccuracy of Eqs. (B10) and (B11). For the values $M\tau = 10^{-4}$ and $N\tau = 10^{-3}$ this inaccuracy is less than 2%.

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[3] V. A. Ditkin and P. I. Kuznetsov, *Handbook of Operational Calculus*, Moscow-Leningrad, 1951.

Translated by R. T. Beyer
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