The (d, p) Reaction on Heavy Nuclei

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The differential cross section for the (d, p) reaction on a heavy nucleus is calculated for the case where the effect of the Coulomb field is fundamental in determining the angular distribution. The differential cross section over the region of large angles increases with increasing angle, while the behavior of the cross section is weakly dependent on the angular momentum l_n of the state into which the neutron is captured. The total cross section is determined and the factor multiplying the exponential is estimated.

AT the present time, the (d, p) reaction has been studied in detail mainly on light nuclei, where the effect of the Coulomb field of the nucleus is not very important. In this case, the protons which are formed emerge mainly in the forward direction and their angular distribution depends strongly on the angular momentum l_n of the state into which the neutron is captured. These experimental data are in good agreement with a formula proposed by Butler[1]; however, it is still not clear why this agreement occurs, since the approximations used by Butler actually correspond to the Born approximation[2], whose applicability to the calculation of the (d, p) reaction has no basis whatever.

We shall consider a different case of the (d, p) reaction—on heavy nuclei, for low energy E_d of the incident deuterons $(E_d < Ze^2/R_0)$, where the effect of the Coulomb field of the nucleus is the main factor in determining the angular distribution. In this case the protons from the (d, p) reaction emerge mainly backward, and their angular distribution depends weakly on l_n. In our case, the whole calculation can be made consistently on the basis of the methods of perturbation theory, which is known to be applicable for large Z and small E_d, because of the smallness of the matrix elements containing the Coulomb wave functions in the repulsive field.

The total cross section for the (d, p) reaction on heavy nuclei (i.e., the Oppenheimer-Phillips process) was calculated by perturbation methods in a whole series of older papers[3-6,7], whose authors limited their considerations to just the spherically symmetric part of the functions $\psi_{\mathbf{k}_d}(\mathbf{r})$ and $\psi_{\mathbf{k}_p}^{(-)}(\mathbf{r})$, describing the motion (in the field of the nucleus) of the incident deuterons and emerging protons, with momenta $\hbar\mathbf{k}_d$ and $\hbar\mathbf{k}_p$ at infinity. To calculate the angular distribution, it is necessary to consider all terms in the expansion of these functions in spherical waves. This is the basic difference between our calculations and those of Lifshitz[7].

1. THE AMPLITUDE FOR THE (d, p) REACTION

The exact value f_{ex} of the amplitude for the reaction $d + A \to B + p$ is given by the formula

$$f_{ex} = -\frac{1}{4\pi}\frac{2M}{\hbar^2}\int \Psi_B^*(R, \mathbf{r}_n)\psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r}_p)[V_{np} + V_p']\Psi_{\mathbf{k}_d}^{(ex)}(R, \mathbf{r}_n, \mathbf{r}_p)dR\,d\mathbf{r}_n d\mathbf{r}_p \quad (1)$$

(the derivation of this formula by the usual methods is given in an Appendix at the end of the paper). Here $V_{np} = V_{np}(|\mathbf{r}_n - \mathbf{r}_p|)$ is the interaction energy of a neutron and a proton whose coordinates are $\mathbf{r}_n$ and $\mathbf{r}_p$, $V_p' = V_p(\mathbf{r}_p, R) - Ze^2/r_p$ is the energy of interaction of the proton with the nucleons of

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[1] S. T. Butler, Proc. Roy. Soc. (London) **208A**, 559 (1951).

[2] E. Gerjuoy, Phys. Rev. **91**, 645 (1953).

[3] J. R. Oppenheimer and M. Phillips, Phys. Rev. **48**, 500 (1935).

[4] H. A. Bethe, Phys. Rev. **53**, 39 (1938).

[5] P. L. Kapur, Proc. Roy. Soc. (London) **163A**, 553 (1937).

[6] G. M. Volkoff, Phys. Rev. **57**, 866 (1940).

[7] E. M. Lifshitz, J. Exper. Theoret. Phys. USSR **8**, 930 (1938).

nucleus A minus the Coulomb energy, R represents the collection of coordinates of the nucleons of nucleus A, at whose center of mass the origin is located, Ψ_B is the wave function of the final nucleus B, $\psi_{\mathbf{k}_p}^{(-)}(\mathbf{r}_p)$ is the Coulomb wave function describing the motion of the proton; as $r_p \to \infty$, it reduces to a sum of an incident plane wave (corresponding to the momentum $\hbar \mathbf{k}_d$) and an outgoing spherical wave*. $\Psi_{\mathbf{k}_d}^{(ex)}$ denotes the exact wave function of the system; to go over from the exact formula (1) to the formula given by the first approximation of perturbation theory, we replace it by the "incident" wave, i.e., by

$$\Psi_A(R)\,\varphi_{\mathbf{k}_d}(\mathbf{r}, \vec{\rho});$$

$$\mathbf{r} = \tfrac{1}{2}(\mathbf{r}_n + \mathbf{r}_p),\quad \vec{\rho} = \mathbf{r}_n - \mathbf{r}_p.$$

Here Ψ_A is the eigenfunction of the nucleus A, while $\varphi_{\mathbf{k}_d}$ describes the motion of the deuteron in the pure Coulomb field Ze^2/r_p:

$$\left\{\left[-\frac{\hbar^2}{4M}\nabla_r^2 + \frac{Ze^2}{|\mathbf{r} - \tfrac{1}{2}\vec{\rho}\,|}\right] + \left[-\frac{\hbar^2}{M}\nabla_\rho^2 + V_{np}(\rho)\right] - (E_d - \varepsilon_d)\right\}\varphi_{\mathbf{k}_d}(\mathbf{r}, \vec{\rho}) = 0,$$

with momentum $\hbar \mathbf{k}_d$ at infinity ($E_d = \hbar^2 k_d^2/4M$, $\epsilon_d = 2.23$ mev is the binding energy of the deuteron). Replacing $\Psi_{\mathbf{k}_d}^{(ex)}$ by $\Psi_A\,\varphi_{\mathbf{k}_d}$ and neglecting in (1) the potential V_p', which differs from zero only for $r_p \leq R_0$ --in the interior of the nucleus A, where $\psi_{\mathbf{k}_p}^{(-)}$ and $\varphi_{\mathbf{k}_d}$ are exponentially small (if Z is large and E_d small), we obtain $f_{ex} \to f_1$, where

$$f_1 = -\frac{1}{4\pi}\frac{2M}{\hbar^2}\int \Phi_n^*(\mathbf{r} + \tfrac{1}{2}\vec{\rho})\,\psi_{\mathbf{k}_p}^{(-)*} \times (\mathbf{r} - \tfrac{1}{2}\vec{\rho})\,V_{np}(\rho)\,\varphi_{\mathbf{k}_d}(\mathbf{r}, \vec{\rho})\,d\mathbf{r}\,d\vec{\rho}. \qquad (2)$$

* We always normalize the wave functions of the continuous spectrum to unit amplitude at infinity.

The function

$$\Phi_n(\mathbf{r}_n) = \int \Psi_A^*(\mathbf{R})\,\Psi_B(R, \mathbf{r}_n)\,dR \qquad (3)$$

is the wave function of the neutron in the final state. This same value of f_1 can be obtained by the standard methods of perturbation theory as the matrix element for the transition between the nonorthogonal states $\Psi_A\varphi_{\mathbf{k}_d}$ and $\Psi_B\,\psi_{\mathbf{k}_p}^{(-)}$.

In the region of values of r_p and r_n greater than R_0, which is the important region in the integrals (1) and (2), $\Psi_{\mathbf{k}_d}^{(ex)}$ is a sum of an "incident" wave $\Psi_A\varphi_{\mathbf{k}_d}$ and various scattered waves (caused solely by nuclear interaction*), whose amplitude decreases exponentially with increasing Z and decreasing E_d. Therefore, the difference $f_{ex} - f_1$ is smaller [i.e., Eq. (2) is more exact] and the perturbation series converges the more rapidly, the larger Z becomes and the smaller E_d.

We express (2) in a form suitable for computation. The regions which are important in the integral (2) are $r > R_0$, and small values $\rho < r_0$, where r_0 is the range of the potential V_{np}; i.e., the important region is $\tfrac{1}{2}\rho \ll r$ (r_0 is always much smaller than R_0). In this region the variables r and ρ in the equation for $\varphi_{\mathbf{k}_d}$ are separable, and its solution to terms of order $(\rho/2r)^2 < (r_0/2R_0)^2$ has the simple form: $\varphi_{\mathbf{k}_d} \approx \varphi_d(\rho)\,\psi_{\mathbf{k}_d}(\mathbf{r})$, where φ_d is the internal wave function of the deuteron:

$$\varphi_d = \sqrt{\frac{\xi_d \varkappa_d}{2\pi}}\,\frac{1}{\rho}\,e^{-\varkappa_d \rho} \quad \text{for } \rho > r_0,$$

($\kappa_d = (M\epsilon_d)^{1/2}/\hbar$, $\xi_d \simeq 3/2$ is a correction factor for normalization) and $\psi_{\mathbf{k}_d}$ is the solution of the equation

$$\left(-\frac{\hbar^2}{2M}\nabla^2 + \frac{Ze^2}{r} - E_d\right)\psi_{\mathbf{k}_d} = 0,$$

* The scattering of the deuteron in the Coulomb field is already taken into account in the "incident" wave $\Psi_A\,\varphi_{\mathbf{k}_d}$; for this reason the word "incident" is given in quotation marks.

which, for $r \to \infty$, goes over into a sum of plane and spherical outgoing waves. In other words, in the region of small $\rho/2r$, which is important in the integral (2), the polarization of the deuteron by the Coulomb field is unimportant*. Substituting this value of $\varphi_{\mathbf{k}_d}$ in (2), and setting $\psi^{(-)}_{\mathbf{k}_p}(\mathbf{r} - \frac{1}{2}\vec{\rho}) \approx \psi^{(-)}_{\mathbf{k}_p}(\mathbf{r})$, $\Phi_n(\mathbf{r} + \frac{1}{2}\vec{\rho}) \approx \Phi_n(\mathbf{r})$, which are correct up to terms in the squares of the quantities $k_p r_0/2$ and $\kappa_n r_0/2$ ($\kappa_n = (2M|E_n|)^{1/2}/\hbar$, $|E_n|$ is the neutron binding energy in nucleus B), which are small compared to unity**, we obtain

$$f_1 = 2\sqrt{\frac{\xi_d \varkappa_d}{2\pi}} \int \Phi_n^*(\mathbf{r})\, \psi^{(-)*}_{\mathbf{k}_p}(\mathbf{r})\, \psi_{\mathbf{k}_d}(\mathbf{r})\, d\mathbf{r}, \qquad (4)$$

since

$$-\frac{1}{4\pi}\frac{2M}{\hbar^2}\int V_{np}(\rho)\,\varphi_d(\rho)\,d\vec{\rho} \approx \sqrt{\frac{\xi_d \varkappa_d}{2\pi}}.$$

A formula completely analogous to (4) was obtained by Landau and Lifshitz[8], in considering the (d, np) reaction (breakup of the deuteron in the field of the nucleus). In this reaction the final state of the neutron is a state of free motion, so that Ψ_B can be represented approximately[8] in the form $\Psi_A \sqrt{\frac{d\mathbf{k}_n}{(2\pi)^3}}\, e^{i\mathbf{k}_n\cdot\mathbf{r}_n}$, i.e., according to (3) $\Phi_n(\mathbf{r}_n)$ is a plane wave $\sim e^{i\mathbf{k}_n\cdot\mathbf{r}_n}$ (if we do not take into account the change of the neutron wave function under the action of the nuclear force field, which is actually not small).

2. THE NEUTRON WAVE FUNCTION $\Phi_n(\mathbf{r})$

In the region $r > R_0$, which is important in (4), the function $\Phi_n(r)$ of Eq. (3) satisfies the equation:

$$-(\hbar^2/2M)\nabla^2\Phi_n = E_n\Phi_n,$$

where $E_n = W_B - W_A$ is the binding energy of the neutron in nucleus B (W_B and W_A are the energies of nuclei B and A), and in accordance with the transformation properties of Ψ_A and Ψ_B under rotation and inversion of the coordinate axes, has the form:

$$\Phi_n(\mathbf{r}, \xi_n) \qquad (5)$$

$$= \sum_{j_n=|J_A-J_B|}^{J_A+J_B} C^{J_B M_B}_{J_A M_A j_n \mu_n} \sum_{\sigma_n=\pm 1/2} C^{j_n \mu_n}_{s_n \sigma_n l_n m_n} \chi_{s_n}$$

$$\times\ \sigma_n(\xi_n)\,\Phi_{j_n l_n m_n}(\mathbf{r}),$$

$$\Phi_{j_n l_n m_n}(\mathbf{r}) = \eta_{j_n l_n}\frac{v_{l_n}(\varkappa_n r)}{r} e^{-\varkappa_n r} Y_{l_n m_n}(\vartheta\varphi),$$

$$v_{l_n} = \sum_{\nu=0}^{l_n} \frac{(l_n+\nu)!}{\nu!\,(l_n-\nu)!}\frac{1}{(2\varkappa_n r)^\nu}.$$

[From now on we write $\Phi_n(\mathbf{r})$ as $\Phi_n(\mathbf{r}, \xi_n)$, i.e., we include the neutron spin variable.] Here $J_A M_A$, $J_B M_B$ are the angular momenta of nuclei A and B, $j_n \mu_n$, $l_n m_n$ and $s_n = \frac{1}{2}$, σ_n are the quantum numbers of the total, orbital and spin angular momenta of the neutron (for given j_n, l_n is $j_n + \frac{1}{2}$ or $j_n - \frac{1}{2}$, depending on the parity of Ψ_A and Ψ_B), $C^{j\mu}_{j_1\mu_1 j_2\mu_2}$

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* In papers 3-6 the amplitude (2) was written in the form

$$f_1 = -\frac{1}{4\pi}\frac{2M}{\hbar^2}\int \Psi_B^*(R, \mathbf{r}_n)$$

$$\psi^{(-)*}_{k_p}(\mathbf{r}_p)\, V_n(\mathbf{r}_n, R)\, \Psi_A(R)\, \varphi_{\mathbf{k}_d}(\mathbf{r}, \vec{\rho})\, d\mathbf{r}_n\, d\mathbf{r}_p,$$

so that the important regions in the integral are $r_n \le R_0$, $r_p > R_0$, i.e., $\rho/2 \approx r$. In this region we have to take into account the deformation of φ_d under the influence of the Coulomb field, and $\varphi_{\mathbf{k}_d}$ has a very complicated structure; the authors of the papers cited limited themselves to the setting up of the spherically symmetric part of $\varphi_{\mathbf{k}_d}$.

** The linear terms in the expansions of $\psi^{(-)}_{\mathbf{k}_p}(\mathbf{r} - \frac{1}{2}\vec{\rho})$ and $\Phi_n(\mathbf{r} = \frac{1}{2}\vec{\rho})$ in powers of ρ vanish when we integrate over the direction of $\vec{\rho}$ in (2). The quadratic terms are small: $(k_p r_0/2)^2 \sim E_p(\text{mev})/40$, $(\kappa_n r_0/2)^2 \sim E_n(\text{mev})/40$.

[8] L. D. Landau and E. M. Lifshitz, J. Exper. Theoret. Phys. USSR **18**, 750 (1948).

are the Clebsch-Gordon coefficients, $\chi_{s_n\sigma_n}$ is the spin function of the neutron, $\kappa_n = (2M|E_n|)^{1/2}/\hbar$ for $E_n < 0$ and $\kappa_n = -ik_n = -i(2ME_n)^{1/2}/\hbar$ for $E_n > 0$. $\eta_{j_n l_n}$ in (5) is a constant depending on the structure of the nucleus. It can be estimated most simply as follows. We first consider those levels of nucleus B for which the neutron energy E_n is positive. If the function (5) is properly normalized (in accordance with the normalization of Ψ_B so that the integral of $|\Psi_B|^2$ over the interior of the nucleus is unity) the total radial flux of neutrons is

$$\frac{\hbar k_n}{M}\frac{1}{2J_B+1}\sum_{M_A M_B}\int |r\Phi_n(\mathbf{r})|^2_{r\to\infty}d\Omega$$

$$=\frac{\hbar k_n}{M}\sum_{j_n}|\eta_{j_n l_n}|^2$$

and is equal to the neutron width $\Gamma_n/\hbar = \sum_{j_n}\Gamma_{j_n l_n}/\hbar$. Thus,

$$|\eta_{j_n l_n}|^2 = (M/\hbar^2 k_n)\,\Gamma_{j_n l_n}.$$

We know[9] that

$$\Gamma_{j_n l_n}$$

$$= 2k_nR_0\,|v_{l_n}(-ik_nR_0)\,e^{ik_nR_0}|^{-2}\,[-f'_{j_n l_n}(W_B)]^{-1},$$

where the function in the square brackets depends on the nuclear structure (W_B is the energy of nucleus B). In order to estimate it, we consider the special case $j_n = 1/2$, $l_n = 0$, when[9,10] $\Gamma_{1/2\,0} = (1/2\pi)(E_n/E_0)^{1/2}\,D_{1/2\,0}(W_B)$, where D is the level spacing and $E_0 \approx 0.7$ mev. Comparing this value with the general expression given above, and setting $\hbar/\sqrt{2ME_0} = \xi_{1/2\,0}/\kappa_d$ [where $\xi_{1/2\,0} = (\epsilon_d/2E_0)^{1/2} \approx 1.25$] for convenience in writing, we have:

$$[-f'_{1/2,0}(W_B)]^{-1} = \frac{\xi_{1/2,0}}{4\pi\varkappa_d}D_{1/2,0}(W_B).$$

We keep this same estimate for $l_n \neq 0$, setting

$$[-f'_{j_n l_n}(W_B)]^{-1} = \frac{\xi_{j_n l_n}}{2\pi\varkappa_d}D_{j_n l_n}(W_B),$$

where $\xi_{j_n l_n}$ is a constant of the order of unity. Then

$$\Gamma_{j_n l_n} \tag{6a}$$

$$=\frac{k_n\xi_{j_n l_n}}{2\pi\varkappa_d\,|v_{l_n}(-ik_nR_0)\exp\{ik_nR_0\}|^2}D_{j_n l_n}(W_B),$$

$$|\eta_{j_n l_n}|^2 = \frac{\varkappa_d\Gamma_{j_n l_n}}{\sqrt{2\epsilon_d E_n}} = \frac{\xi_{j_n l_n}\varkappa_d}{2\pi}$$

$$\times|v_{l_n}(-ik_nR_0)\exp\{ik_nR_0\}|^{-2}\frac{D_{j_n l_n}(W_B)}{\epsilon_d}.$$

Thus, the condition that the integral of $|\Psi_B|^2$ over the interior of the nucleus be equal to unity determines $|\eta_{j_n l_n}|^2$ as an analytic function of the neutron energy E_n. The form of this function does not change when we go from levels of nucleus B with $E_n > 0$ to levels with $E_n < 0$; when this is done we must write κ_n in place of $-ik_n$ on the right-hand side of (6a)*:

$$|\eta_{j_n l_n}|^2 = \frac{\xi_{j_n l_n}\varkappa_d}{2\pi} \tag{6b}$$

$$\times\ |v_{l_n}(\varkappa_nR_0)\exp\{-\varkappa_nR_0\}|^{-2}\cdot\frac{D_{j_n l_n}(W_B)}{\epsilon_d}.$$

Formulas (5)-(6) completely determine $\Phi_n(\mathbf{r})$ over the region which is important in the integral (4).

[9] H. Feshbach, D. R. Peaslee and V. F. Weisskopf, Phys. Rev. **71**, 145 (1947).

[10] A. I. Akhiezer and I. Ia. Pomeranchuk, *Some Problems of Nuclear Theory*, GITTL, 1950.

* This same value (6b) can be gotten directly from the condition that the integral of $|\Psi_B|^2$ over the interior of the nucleus be equal to unity. We emphasize once more that the constants κ_d and ϵ_d of the theory of the deuteron are introduced here only for convenience in writing the formulas.

3. THE DIFFERENTIAL CROSS SECTION

Since the spin of the neutron was included in (5) to make our formulas precise, we shall give the expression (4) for the amplitude including the spins of all the particles:

$$f_1 = \sqrt{\frac{2\xi_d \varkappa_d}{\pi}} \int \left(\Phi_n^*(\mathbf{r}, \xi_n)\, \psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r}) \cdot \chi_{s_p \sigma_p}^*(\xi_p),\right.$$

$$\left.\psi_{\mathbf{k}_d}(\mathbf{r})\, \chi_{s_d \sigma_d}(\xi_n \xi_p)\right) d\mathbf{r},$$

where $\chi_{s_p \sigma_p}$ and $\chi_{s_d \sigma_d}$ are the spin functions of the proton and deuteron ($s_p = 1/2$, $s_d = 1$). Substituting Φ_n from (5), we express the differential cross section for the (d, p) reaction

$$d\sigma = \frac{1}{(2s_d+1)(2J_A+1)} \sum_{M_A \sigma_d,\, M_B \sigma_p} |f_1|^2 \frac{v_p}{v_d} d\Omega_p,$$

$$\left(\frac{v_p}{v_d} = \frac{2k_p}{k_d}\right)$$

in the form

$$d\sigma = \frac{2J_B+1}{2(2J_A+1)} \sum\nolimits_{j_n} d\sigma_{j_n l_n}, \tag{7}$$

$$d\sigma_{j_n l_n} = \frac{1}{2l_n+1} \sum_{m_n=-l_n}^{l_n} |f_{l_n m_n}^{(1)}(\vartheta_p)|^2 \frac{2k_p}{k_d} d\Omega_p;$$

$$f_{l_n m_n}^{(1)}(\vartheta_p) = \sqrt{\frac{2\xi_d \varkappa_d}{\pi}}\, \eta_{j_n l_n} \tag{8}$$

$$\times \int v_{l_n}(\varkappa_n r)\, e^{-\varkappa_n r}\, Y_{l_n m_n}^*(\vartheta\varphi)\, \psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r})\, \psi_{\mathbf{k}_d}(\mathbf{r}) \frac{d\mathbf{r}}{r}.$$

Here we have made use of the facts that

$$\sum_{\sigma_p \sigma_d} \Big| \sum_{\sigma_n} A_{\sigma_n} (\chi_{s_n \sigma_n}^* \chi_{s_p \sigma_p}^*,\ \chi_{s_d \sigma_d}) \Big|^2 = \frac{3}{2} \sum_{\sigma_n} |A_{\sigma_n}|^2,$$

$$\sum_{(M_B - M_A = \mu_n)}' C_{J_A M_A,\, j_n \mu_n}^{J_B M_B} C_{J_A M_A,\, j_n' \mu_n}^{J_B M_B} = \frac{2J_B+1}{2j_n+1} \delta_{j_n j_n'};$$

$$\sum_{(\mu_n - \sigma_n = m_n)}' C_{s_n \sigma_n,\, l_n m_n}^{j_n \mu_n} C_{s_n \sigma_n,\, l_n' m_n}^{j_n \mu_n} = \frac{2j_n+1}{2l_n+1} \delta_{l_n l_n'}.$$

The A_{σ_n} are arbitrary quantities: $\sum'_{(\mu_n - \sigma_n = m_n)}$ denotes summation over all values of μ_n and σ_n for which $\mu_n - \sigma_n = m_n$, where m_n is fixed. We note that formulas (7)-(8) correspond to the formulas of Butler.

We omit from the integral (8) the region of integration $r \leq R_0$, which is unimportant in the case we are considering, and set:

$$f_{l_n m_n}^{(1)} = \sqrt{\frac{2\xi_d \varkappa_d}{\pi}} \tag{8a}$$

$$\times \int_{(r > R_0)} \Phi_{j_n l_n m_n}^*(\mathbf{r})\, \psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r})\, \psi_{\mathbf{k}_d}(\mathbf{r})\, d\mathbf{r}.$$

For large Z, (8a) practically coincides with (8); on the other hand, if we formally set $Z = 0$ in (8a), so that $\psi_{\mathbf{k}_p}^{(-)*} \psi_{\mathbf{k}_d} = \exp\{i(\mathbf{k}_d - \mathbf{k}_p) \cdot \mathbf{r}\}$, then (8a) and (5) give Butler's[1] result:

$$f_{l_n m_n}^{(1)} = \sqrt{\frac{\xi_d \varkappa_d}{2\pi}}\, \eta_{j_n l_n} \frac{4\pi i^{l_n} R_0^2 Y_{l_n m_n}^*(\vartheta_q \varphi_q)}{\varkappa_d^2 + (1/2\mathbf{k}_p - \mathbf{k}_d)^2}$$

$$\times \left[G_{l_n}(qr) \frac{d}{dr}\left(\frac{v_{l_n}(\varkappa_n r)}{r} e^{-\varkappa_n r}\right) - \frac{dG_{l_n}(qr)}{dr} \frac{v_{l_n}(\varkappa_n r)}{r} e^{-\varkappa_n r} \right]_{r=R_0},$$

where $q = |\mathbf{k}_d - \mathbf{k}_p|$, $G_{l_n}(x) = \sqrt{\pi/2x}\, J_{l_n + 1/2}(x)$, $J_{l_n+1/2}$ is a Bessel function.

Thus (8a) is an interpolation formula, correct for large Z and reducing to Bulter's formula in the limit $Z \to 0$. In the following we shall limit ourselves to the case of large Z and shall use (8) rather than (8a), with the value of Φ_n given by (5) even for $r \leq R_0$, since it is more convenient to calculate the integral over the whole space. Thus the results which we shall obtain do not reduce to Butler's for $Z \to 0$.

4. ANGULAR DISTRIBUTION

In the quasiclassical case which we are considering, when Z is large and E_d small, and $\alpha_d = Ze^2/\hbar v_d > 1$, $\alpha_p = Ze^2/\hbar v_p > 1$ ($v_p = \sqrt{2E_p/M}$, $v_d = \sqrt{E_d/M}$), the angular distribution and the variation of the cross section with energy depend weakly on l_n. In fact, in this case the factor

$$e^{-\varkappa_n r}\psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r})\,\psi_{\mathbf{k}_d}(\mathbf{r}) = \exp\{-\varkappa_n r + \ln[\psi_{\mathbf{k}_p}^{(-)*}\psi_{\mathbf{k}_d}]\}$$

in the integral (8) varies exponentially as a function of $\mathbf{r}$.

The value of the integral of such a rapidly varying function is determined by its value in the neighborhood of the saddle point $\mathbf{r}_1(r_1, \vartheta_1, \varphi_1)$, at which $F(\mathbf{r}) = -\kappa_n r + \ln[\psi_{\mathbf{k}_p}^{(-)*}\psi_{\mathbf{k}_d}]$ is an extremum. Therefore, the slowly varying spherical functions in (8) can be evaluated at $\vartheta = \vartheta_1$ and $\varphi = \varphi_1$ and removed from the integral; if* $\kappa_n r_1 > l_n(l_n+1)/2$ then the function $v_{l_n} = 1 + l_n(l_n+1)/2\kappa_n r + \ldots$ in the remaining integral can be replaced by its asymptotic value $v_{l_n} \approx 1$. This gives

$$f_{l_n m_n}^{(1)} = \sqrt{2\xi_d\varkappa_d/\pi}\,\eta_{j_n l_n}\sqrt{4\pi}\,Y_{l_n m_n}(\vartheta_1, \varphi_1)\,I_0(\vartheta_p), \tag{9}$$

$$I_0(\vartheta_p) = \frac{1}{\sqrt{4\pi}}\int e^{-\varkappa_n r}\psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r})\,\psi_{\mathbf{k}_d}(\mathbf{r})\frac{d\mathbf{r}}{r}.$$

Substituting $f_{l_n m_n}^{(1)}$ in (7) and using the fact that $\frac{4\pi}{2l_n+1}\sum_{m_n=-l_n}^{l_n} |Y_{l_n m_n}(\vartheta_1, \varphi_1)|^2 = 1$, we get

$$d\sigma_{j_n l_n} = a_{j_n l_n}\varkappa_d^2\,|I_0(\vartheta_p)|^2\frac{2k_p}{k_d}\,d\Omega_p. \tag{10}$$

Here $a_{j_n l_n}$ denotes a dimensionless constant, independent of ϑ_p and E_d and equal, according to (6a, b), to:

$$a_{j_n l_n} = \frac{2\xi_d}{\pi}\frac{|\eta_{j_n l_n}|^2}{\varkappa_d} \tag{11}$$

$$= \begin{cases} \dfrac{\sqrt{2}\xi_d}{\pi}\dfrac{\Gamma_{j_n l_n}}{\sqrt{\varepsilon_d E_n}}, & E_n > 0 \\[2ex] \dfrac{\xi_d \xi_{j_n l_n}}{\pi^2}\dfrac{D_{j_n l_n}(W_B)\exp\{2\varkappa_n R_0\}}{\varepsilon_d\,|v_{l_n}(\varkappa_n R_0)|^2}, & E_n < 0. \end{cases}$$

Thus only the absolute value of the cross section depends on l_n, and not its variation with angle and energy.

After substituting in (9) the exact values of the Coulomb functions:

$$\psi_{\mathbf{k}_d} = \sqrt{2\pi\alpha_d}\exp\{-\pi\alpha_d + i\mathbf{k}_d\cdot\mathbf{r}\}\,F(-i\alpha_d, 1, i(k_d r - \mathbf{k}_d\cdot\mathbf{r})),$$

$$\psi_{\mathbf{k}_p}^{(-)*} = \sqrt{2\pi\alpha_p}\exp\{-\pi\alpha_p - i\mathbf{k}_p\cdot\mathbf{r}\}\,F(-i\alpha_p, 1, i(k_p r + \mathbf{k}_p\cdot\mathbf{r}))$$

the integral can be computed exactly (similar integrals were evaluated by Sommerfeld[11,12]). The details are given in the Appendix, where it is shown that

$$I_0(\vartheta_p) = \frac{4\pi^{3/2}\sqrt{\alpha_d\alpha_p}}{(k_d-k_p)^2+\varkappa_n^2}\exp\{-\pi(\alpha_d+\alpha_p)\} \times \left[\frac{(\varkappa_n - ik_d)^2 + k_p^2}{\varkappa_n^2 + (k_d-k_p)^2}\right]^{i\alpha_d} \times \left[\frac{(\varkappa_n - ik_p)^2 + k_d^2}{\varkappa_n^2 + (k_d-k_p)^2}\right]^{i\alpha_p}\frac{F(-i\alpha_d, -i\alpha_p, 1, -\zeta)}{1+\zeta}.$$

The argument ζ of the hypergeometric function depends on the angle ϑ_p between $\mathbf{k}_d$ and $\mathbf{k}_p$:

---

* This condition is almost always fulfilled if l_n is not very large and $|E_n|$ is not very small. For an estimate, we can use the value $r_1 = Ze^2/[E_d + (\sqrt{2|E_n|} - \sqrt{\varepsilon_d})^2]$, which was gotten[7] in calculating an integral of the type of (9) by the saddle-point method. Then

$$\varkappa_n r_1 = 0{,}15Z\frac{\sqrt{2\varepsilon_d|E_n|}}{E_d + (\sqrt{2|E_n|} - \sqrt{\varepsilon_d})^2} \quad \left(0{,}15 = \frac{e^2}{\hbar}\sqrt{\frac{M}{\varepsilon_d}}\right),$$

which, for heavy nuclei, gives a value of about ten for $\kappa_n r_1$.

[11] A. Sommerfeld, *Atombau und Spectrallinien IIB.*, Braunschweig, 1939.

[12] A. Sommerfeld, Ann. d. Physik 11, 257 (1931).

$$\zeta = \zeta_0 \sin^2(\vartheta_p/2);\quad \zeta_0 = \frac{4k_d k_p}{(k_d - k_p)^2 + \varkappa_n^2} = \frac{4\sqrt{2E_d(E_d+Q)}}{(\sqrt{2E_d} - \sqrt{E_d+Q})^2 + E_n},$$

where $Q = E_p - E_d = -E_n - \epsilon_d$ is the Q of the reaction. For $E_n < 0$, κ_n is real, so that the quantities in square brackets in $I_0(\vartheta_p)$ are complex:

$$(\varkappa_n - ik_d)^2 + k_p^2 = -(k_d^2 - \varkappa_n^2 - k_p^2) - 2ik_d\varkappa_n = c_1 e^{-i(\pi - \varphi_d)},$$

$$(\varkappa_n - ik_p)^2 + k_d^2 = (k_d^2 + \varkappa_n^2 - k_p^2) - 2ik_p\varkappa_n = c_2 e^{-i\varphi_p},$$

where c_1 and c_2 are the moduli of the complex quantities, and the phases are determined (for $E_n < 0$) by the equations:

$$\operatorname{tg}\varphi_d = \frac{2\varkappa_n k_d}{k_d^2 - k_p^2 - \varkappa_n^2} = \frac{2\sqrt{2|E_n|E_d}}{E_d + \varepsilon_d - 2|E_n|}, \quad 0 \leqslant \varphi_d \leqslant \pi, \tag{12a}$$

$$\operatorname{tg}\varphi_p = \frac{2\varkappa_n k_p}{k_d^2 - k_p^2 + \varkappa_n^2} = \frac{2\sqrt{|E_n|(E_d + Q)}}{E_d + \varepsilon_d}, \quad 0 \leqslant \varphi_p \leqslant \frac{\pi}{2}.$$

According to (12a), the angles φ_d and φ_p go to zero for $|E_n| \to 0$, and obviously remain equal to zero for any $E_n > 0$, when $\kappa_n = -ik_n$ is a pure imaginary number:

$$E_n > 0,\quad \varphi_d = \varphi_p = 0. \tag{12b}$$

We can therefore write, for $E_n > 0$ and $E_n < 0$:

$$|I_0(\vartheta_p)| = \frac{4\pi\sqrt{\pi\alpha_d\alpha_p}}{(k_d - k_p)^2 + \varkappa_n^2} \exp\{-(\alpha_d\varphi_d - \alpha_p\varphi_p)\} \times \left|\frac{e^{-\pi\alpha_p} F(i\alpha_d, i\alpha_p, 1, -\zeta)}{1+\zeta}\right|$$

or, according to (10):

$$d\sigma_{l_n l_n} = a_{l_n l_n} \frac{8\pi^2\varkappa_d^2\alpha_d}{[(k_d - k_p)^2 + \varkappa_n^2]^2} \times \exp\{-2(\alpha_d\varphi_d - \alpha_p\varphi_p)\} N(\zeta)\, d\Omega_p, \tag{13}$$

where the function

$$N(\zeta) = 2\pi\alpha_d \left|\frac{e^{-\pi\alpha_p} F(i\alpha_d, i\alpha_p, 1, -\zeta)}{1+\zeta}\right|^2 \tag{14}$$

determines the dependence of the cross section on ϑ_p. In the case we are considering, when $\alpha_d > 1$, or $\alpha_p > 1$, the hypergeometric function has the following asymptotic value:

$$|F(i\alpha_d, i\alpha_p, 1, -\zeta)|^2 \approx \frac{1+\zeta}{2\pi\alpha_d\zeta} \frac{\exp\{2\pi\alpha_p + 2(\alpha_d\psi_d - \alpha_p\psi_p)\}}{\sqrt{(4\rho/\zeta) - (1-\rho)^2}}, \tag{15}$$

the derivation of which is given in the Appendix. The angles ψ_d and ψ_p are defined by the equations

$$\cos\psi_d = \frac{(1-\rho)\zeta + 2}{2\sqrt{1+\zeta}},\quad 0 \leqslant \psi_d \leqslant \pi, \tag{16}$$

$$\cos\psi_p = \frac{(1-\rho)\zeta - 2\rho}{2\rho\sqrt{1+\zeta}},\quad 0 \leqslant \psi_p \leqslant \pi, \tag{16a}$$

where $\rho = \alpha_p/\alpha_d = k_d/2k_p = (E_d/2E_p)^{1/2}$. The equality (15) is valid for not too small ζ—roughly speaking, until $\zeta \sim 1/\alpha_d$; more precisely, for $\zeta > \zeta'$, where

$$\frac{\zeta'^2}{1+\zeta'}\sqrt{(4\alpha_d\alpha_p/\zeta') - (\alpha_d - \alpha_p)^2} = \min\left\{\frac{\alpha_p}{\alpha_d}, \frac{\alpha_d}{\alpha_p}\right\}$$

(cf. the Appendix). If $\zeta \to 0$, then $|F(i\alpha_d, i\alpha_p, 1, -\zeta)/(1+\zeta)| \to 1$. Therefore:

$$N(\zeta)=\begin{cases}[\zeta(1+\zeta)\sqrt{(4\rho/\zeta)-(1-\rho)^2}]^{-1}\exp\{2(\alpha_d\psi_d-\alpha_p\psi_p)\}, & \text{if } \zeta>\zeta' \\ 2\pi\alpha_d\exp\{-2\pi\alpha_p\}, & \text{if } \zeta\to 0,\end{cases} \qquad (17)$$

or

$$N(\zeta)=\exp\{\alpha_d L_\rho(\zeta)\}/M_\rho(\zeta), \qquad (18)$$

where

$$L_\rho(\zeta)=2(\psi_d-\rho\psi_p), \quad L_\rho^{\cdot}(0)=-2\pi\rho,$$

$$M_\rho(\zeta)=\zeta(1+\zeta)\sqrt{(4\rho/\zeta)-(1-\rho)^2},$$

$$M_\rho(0)=1/2\pi\alpha_d.$$

Graphs of the functions $L_\rho(\zeta)$ and $M_\rho(\zeta)$ are shown in Fig. 1. As we see from the figure, $L_\rho(\zeta)$ increases with increasing ζ [reaching zero for $\zeta=4\rho/(1-\rho)^2$, while $\zeta_0<4\rho/(1-\rho)^2$ throughout]. This increase causes an exponential increase in $N(\zeta)$, i.e., in the cross section, with increasing ϑ_p.

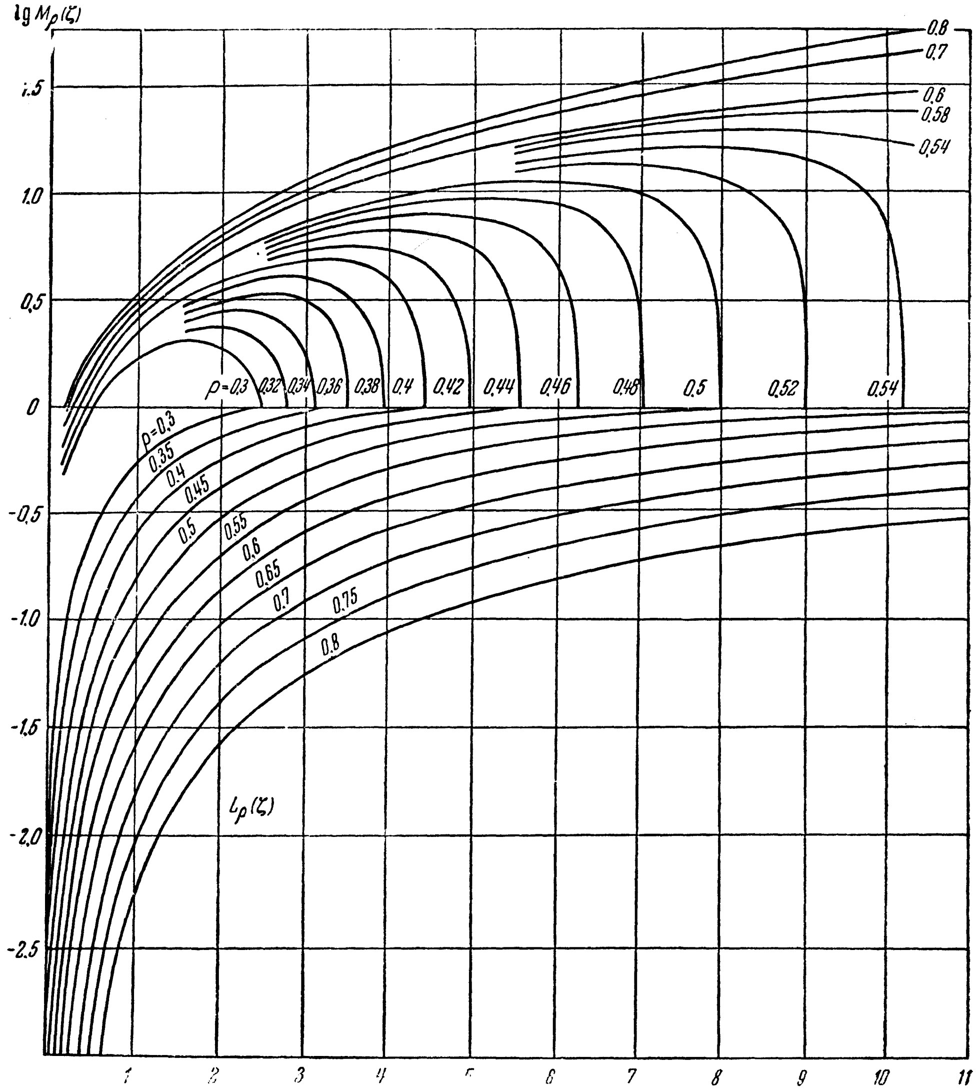

FIG. 1

The function $N(\zeta)$ is a maximum for $\vartheta_p=\pi$, $\zeta=\zeta_0$; for small $\pi-\vartheta_p$, its dependence on $\pi-\vartheta_p$ is approximately a Gaussian. This can be seen by expanding $L_\rho(\zeta)$ in series in powers of $\zeta_0-\zeta=\zeta_0\cos^2(\vartheta_p/2)\approx\zeta_0(\pi-\vartheta_p)^2/4$:

$$L_\rho(\zeta) = L_\rho(\zeta_0) - \frac{\varkappa_d}{k_p(1+\zeta_0)}(\zeta_0 - \zeta) + \cdots$$

(since $\left(\frac{dL_\rho}{d\zeta}\right)_{\zeta_0} = \frac{\sqrt{(4\rho/\zeta_0) - (1-\rho)^2}}{1+\zeta_0}$, $\sqrt{\frac{4\rho}{\zeta_0} - (1-\rho)^2} = \frac{\varkappa_d}{k_p}\Big]$) and neglecting the change of the factor $M_\rho(\zeta)$ outside of the exponential in (18). We then get:

$$N(\zeta) \simeq A \exp\{-(\pi - \vartheta_p)^2 \delta^{-2}\},$$

$$\pi - \vartheta_p \ll 1,$$

where

$$\delta^2 = \frac{(k_d - k_p)^2 + \varkappa_n^2}{\alpha_d \varkappa_d k_d} = \frac{(\sqrt{2E_d} + \sqrt{E_d + Q})^2 + Q + \varepsilon_d}{\beta \varepsilon_d},$$

$$\beta = \frac{Ze^2}{\hbar}\sqrt{\frac{M}{\varepsilon_d}} = 0.15\, Z.$$

The width δ of the angular distribution is smaller, the larger Z and the smaller E_d, and the higher the nuclear level into which the neutron is captured (i.e., the smaller Q). This conclusion also follows from consideration of the angular distribution curves of Fig. 2 for various cases (the curves are drawn on a logarithmic scale).

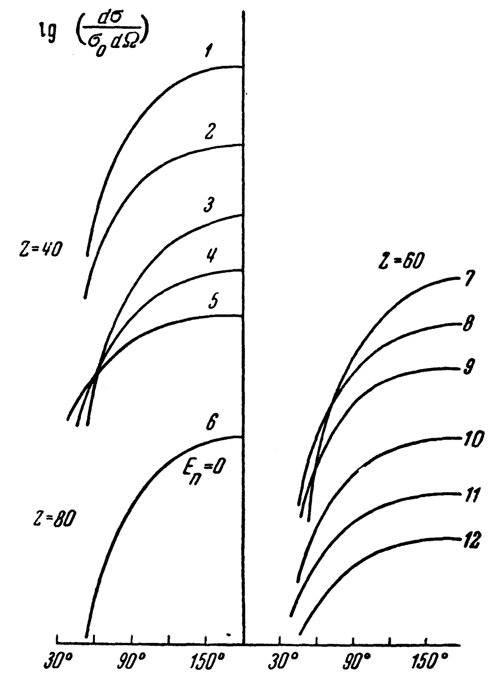

FIG. 2. 1—E_n=—2, 2—E_n=—4, E=2 mev; 3—E_n=1, 4—E_n=0, 5—E_n=—2, E=4 mev; 6—E_n=0, E=6 mev; 7—E_n==0, 8—E_n=—2, 9—E_n=—4, E=4 mev; 10—E_n=0, 11—E_n=—2, 12—E_n=—4, E=6 mev

5. TOTAL CROSS SECTION

In order to calculate the total cross section, we integrate (13) over ϑ_p; noting that $d\Omega_p = (4\pi/\zeta_0)\,d\zeta$, we get the integral:

$$\int N(\zeta)\, d\Omega_p = \frac{4\pi}{\zeta_0}\int_0^{\zeta_0} N(\zeta)\, d\zeta = \frac{4\pi}{\zeta_0}\int_0^{\zeta_0} \frac{\exp\{\alpha_d L_\rho(\zeta)\}\, d\zeta}{\zeta(1+\zeta)\sqrt{\frac{4\rho}{\zeta} - (1-\rho)^2}},$$

in which, because of the decrease of $L_\rho(\zeta)$ with increasing $\zeta_0 - \zeta$, the values of ζ near to ζ_0 are important. Therefore, as before, we expand $L_\rho(\zeta)$ in powers of $x = \zeta_0 - \zeta$ and extend the x integration to infinity. This gives

$$\int N(\zeta)\, d\Omega_p$$

$$= \frac{\pi}{4\alpha_d}\left[\frac{(k_d - k_p)^2 + \varkappa_n^2}{k_d \varkappa_d}\right]^2 \exp\{\alpha_d L_\rho(\zeta_0)\}$$

(the factor in front of the exponential was taken outside the integral and evaluated at $\zeta = \zeta_0$).

According to (13) we then have:

$$\sigma_{j_n l_n} = 2\pi^3 a_{j_n l_n} k_d^{-2} \exp\{-\beta\Phi(E_d, E_n)\}, \qquad (19)$$

$$\beta\Phi(E_d, E_n) = 2(\alpha_d\varphi_d - \alpha_p\varphi_p) - \alpha_d L_\rho(\zeta_0)$$
$$= 2[\alpha_d(\varphi_d - \psi_d^{(0)}) - \alpha_p(\varphi_p - \psi_p^{(0)})]$$

(where $\alpha_d = \beta\sqrt{\epsilon_d/E_d}$, $\alpha_p = \beta\sqrt{\epsilon_d/2E_p}$). $\psi_d^{(0)}$ and $\psi_p^{(0)}$ are the angles (16) for $\zeta = \zeta_0$. Writing

$$\Phi(E_d, E_n) = \eta_d\sqrt{4\varepsilon_d/E_d} - \eta_p\sqrt{2\varepsilon_d/E_p}, \qquad (20a)$$

and using (12a, b)-(16), we express $\eta_d = \varphi_d - \psi_d^{(0)}$ and $\eta_p = \varphi_p - \psi_p^{(0)}$ directly in terms of E_d and E_n:

$$E_n > 0 \begin{cases} \operatorname{tg}\eta_d = \sqrt{2E_d\varepsilon_d}/[E_n - (E_d + Q)], & -\pi \leqslant \eta_d \leqslant 0, \\ \operatorname{tg}\eta_p = 2\sqrt{2(E_d + Q)\varepsilon_d}/[E_n - (E_d + Q) - 2\varepsilon_d], & -\pi \leqslant \eta_p \leqslant 0, \end{cases} \qquad (20b)$$

$$E_n < 0 \begin{cases} \operatorname{tg}\frac{\eta_d}{2} = (\sqrt{2|E_n|} - \sqrt{\varepsilon_d})E_d^{-1/2}, & -\pi \leqslant \eta_d \leqslant \pi. \\ \operatorname{tg}\frac{\eta_p}{2} = (\sqrt{|E_n|} - \sqrt{2\varepsilon_d})(E_d + Q)^{-1/2} & -\pi \leqslant \eta_p \leqslant \pi. \end{cases}$$

These formulas can also be written in the following form ($E_n \gtrless 0$):

$$\Phi(E_d, E_n) = 2\mathrm{Re}\left\{\sqrt{\frac{4\varepsilon_d}{E_d}}\operatorname{arc\,tg}\frac{\sqrt{-2E_n} - \sqrt{\varepsilon_d}}{\sqrt{E_d}} - \sqrt{\frac{2\varepsilon_d}{E_p}}\operatorname{arc\,tg}\frac{\sqrt{-E_n} - \sqrt{2\varepsilon_d}}{\sqrt{E_p}}\right\}$$

$$= 2\,\mathrm{Im}\left\{\sqrt{\frac{4\varepsilon_d}{E_d}}\operatorname{arc\,cosh}\frac{E_d^{1/2}}{[E_d + (\sqrt{\varepsilon_d} - \sqrt{-2E_n})^2]^{1/2}} - \sqrt{\frac{2\varepsilon_d}{E_p}}\operatorname{arc\,cosh}\frac{E_p^{1/2}}{[E_d + (\sqrt{\varepsilon_d} - \sqrt{-2E_n})^2]^{1/2}}\right\}.$$

In this form, $\Phi(E_d, E_n)$ coincides with the expression obtained by Lifshitz[7]. It is not difficult to see that the initial formula of his calculation[7] can be gotten from (8) and (9), if instead of $\psi_{\mathbf{k}_p}^{(-)}$ and $\psi_{\mathbf{k}_d}$, we substitute the spherically symmetric parts of those functions in the quasiclassical approximation, and apply the saddle-point method to evaluate the integral

$$\int_0^\infty \exp\left\{-\beta\sqrt{2\varepsilon_d}\left[\int_{E_d}^{x}\sqrt{2(y - E_d)}\frac{dy}{y^2} - \int_{E_p}^{x}\sqrt{y - E_p}\frac{dy}{y^2} - \frac{\sqrt{-E_n}}{x}\right]\right\}dx,$$

to which (9) then reduces. (x and y are reciprocal to r: $x = Ze^2/r$.)

Figure 3 shows curves of the dependence of $\Phi(E_d, E_n)$ on $(|E_n|/2\epsilon_d)^{1/2}$ for the values of $(E_d/\epsilon_d)^{1/2}$ marked on the individual curves. The curves were constructed for the region $E_n < 0$, and their behavior agrees with the analysis of the behavior of the functions $\Phi(E_d, E_n)$ which was given by Lifshitz.

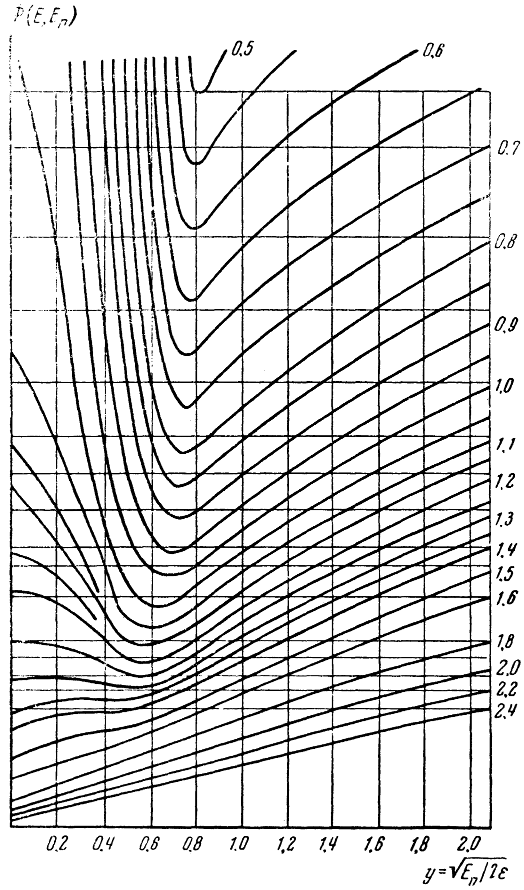

FIG. 3

Substituting (11) in (19), we write the total cross section in the form:

$$\sigma_{j_n l_n} = \begin{cases} \dfrac{\pi^2 \xi_d \Gamma_{j_n l_n}}{\varkappa_d k_n E_d} \exp\{-\beta\Phi_1(E_d, E_n)\} & (E_n > 0) \\ \dfrac{\pi \xi_d \xi_{j_n l_n} D_{j_n l_n}(W_B)}{2\varkappa_d^2 E_d \,|v_{l_n}(\varkappa_n R_0)|^2} \exp\{-\beta\Phi_1(E_d, E_n)\} & (E_n < 0); \end{cases}$$

where $\beta\Phi_1 = \beta\Phi - 2\varkappa_n R_0$, i.e., $\Phi_1 = \Phi - \sqrt{8\epsilon_d |E_n|/B}$, $B = Ze^2/R_0$ is the height of the Coulomb barrier.

Finally, we calculate the overall cross section

$$\sigma_{\mathrm{tot}} = \sum \frac{2J_B + 1}{2(2J_A + 1)} \Sigma_{j_n} \sigma_{j_n l_n} \qquad (22)$$

(summed over all levels of nucleus B with $E_n < 0$) for capture of the neutron into any level with $E_n < 0$ of a heavy nucleus, whose levels are distributed in the region $E_0 \simeq 0$. [Capture in a level with $E_n > 0$ would actually lead to the (d, np) reaction.] Then in the summation in (22) we need only include levels with $l_n = 0$, i.e, $J_B = J_A \pm 1/2$, since $e^{-\beta\Phi_1}$ drops rapidly* with increasing $|E_n|$ so that only levels with small

* We should keep in mind the case when $(E_d/\epsilon_d)^{1/2} \geq 1.8$, where, according to Fig. 3, Φ increases approximately linearly with increasing $|E_n|^{1/2}$.

$|E_n|$ are important in the summation, while for small $|E_n|$, the quantity $|v_{l_n}(\kappa_n R_0)|^{-2}$ is proportional to $E_n^{l_n}$. Going over from a summation over levels to an integration [the sum over levels is equal to $\int d|E_n|/D(W_B)$], and using the fact that on the average $\overline{2J_B+1} = 2J_A + 1$ (if $J_B = J_A \pm ½$), we get from (21):

$$\sigma_{\text{tot}} = \frac{1}{2}\frac{\pi\xi_d\xi_{1/2,0}}{2\varkappa_d^2}\int\limits_0^{|E_n^0|}\exp\{-\beta\Phi_1(E_d, E_n)\}\frac{d|E_n|}{E_d}, \tag{23a}$$

$|E_n^0|$ is the binding energy of the neutron in the ground state of nucleus B. The integral can be calculated approximately, by expanding $\Phi_1(E_d, E_n)$ in powers of $|E_n|^{½}$:

$$\beta\Phi_1(E_d, E_n) = \beta\Phi(E_d, 0) + \sqrt{\frac{8}{E_0}}\frac{B-E_d-\varepsilon_d}{E_d+\varepsilon_d}\sqrt{|E_n|},\quad E_0 = \frac{\hbar^2}{MR_0^2}$$

[according to (20a, b), $(\partial\Phi/\partial|E_n|^{½})_{E_n=0} = \sqrt{8\epsilon_d}/(E_d+\epsilon_d)$] and extending the integration over $|E_n|^{½}$ to infinity. We obtain

$$\sigma_{\text{tot}} = \frac{\pi\xi_d\xi_{1/2,0}}{16\varkappa_d^2}\frac{E_0}{E_d}\left(\frac{E_d+\varepsilon_d}{B-E_d-\varepsilon_d}\right)^2\exp\{-\beta\Phi(E_d, 0)\}, \tag{23b}$$

where, according to formulas given earlier,

$$\Phi(E_d, 0) = 2\left(\sqrt{\frac{2\varepsilon_d}{E_p}}\operatorname{arc\,tg}\sqrt{\frac{2\varepsilon_d}{E_p}} - \sqrt{\frac{4\varepsilon_d}{E_d}}\operatorname{arc\,tg}\sqrt{\frac{\varepsilon_d}{E_d}}\right).$$

6. THE REGION OF VALIDITY OF THE CALCULATION

We must emphasize that the region of applicability of formulas (18)-(23) is very limited. The fundamental reason for this is related to our having neglected all terms Δf proportional to the value of $\psi_{\mathbf{k}_p}^{(-)*}\psi_{\mathbf{k}_d}$ in the region inside the nucleus* compared to the amplitude f_1 given by formulas (4)-(8). According to (13), f_1 is proporportional to the exponential

$$\exp\left\{\frac{\beta}{2}\Phi(E_d, E_n) - \frac{1}{2}\ln\frac{N(\zeta_0)}{N(\zeta)}\right\},$$

while we have dropped terms corresponding to the value of $\psi_{\mathbf{k}_p}^{(-)*}\psi_{\mathbf{k}_d}$ for $r = R_0$, which in the quasiclassical approximation (if we consider only S-waves) is determined by the exponential

$$\exp\{-{}^1/_2\,\beta\Phi_z(E_d, E_n)\},$$

$$\Phi_z = 2\sqrt{\frac{4\varepsilon_d}{B}}\left\{\gamma\left(\frac{E_d}{B}\right) + \frac{1}{\sqrt{2}}\gamma\left(\frac{E_p}{B}\right)\right\},$$

$$\gamma(x) = \sqrt{x}\arccos\sqrt{x} - \sqrt{1-x}.$$

If we assume (as is verified by calculation) that the factors in front of the exponentials in Δf and f_1 are of the same order, then we find that neglecting Δf compared to f_1 is permissible if

$$\beta(\Phi_z - \Phi) - \ln\frac{N(\zeta_0)}{N(\zeta)} > 2. \tag{24}$$

* In particular, we substituted into (8) the incorrect value (5) for Φ_n in the region $r < R_0$. If the correct value is substituted, the region inside the nucleus automatically gives a small contribution to the integral (8), because of the rapid oscillation or the damping of Φ_n in this region.

This condition severely limits the values of Z, E_d and ϑ_p. It is well satisfied only for heavy nuclei ($Z > 50$), for $E_d < B$, and under conditions where the angle ϑ_p is not small. Thus, even for heavy nuclei, formulas (17)-(18) give the proton angular distribution correctly only in the region of large values of ϑ_p. This significantly decreases the reliability of the data given in Sec. 5 on total cross sections; for more accurate calculations, we would have to start not from (8), but from (8b), where the region inside the nucleus is taken into account completely.

The author expresses his thanks to Academician L. D. Landau for discussion and many valuable comments. The present calculation was carried out in 1951, in connection with experimental work of Academician P. I. Lukirskii and Prof. Iu. A. Nemilov.

APPENDIX

I. DERIVATION OF FORMULA (1) FOR f_{ex}

We multiply both sides of the Schrodinger equation:

$$\Big\{-\frac{\hbar^2}{2M}\nabla^2_{r_p}+\Big[-\frac{\hbar^2}{2M}\nabla^2_{r_n}+V_n(\mathbf{r}_n, R)+\hat{\mathscr{H}}_A(R)\Big] + V_p(\mathbf{r}_p, R) + V_{np}(|\mathbf{r}_n-\mathbf{r}_p|)\Big\}\Psi^{(ex)}_{\mathbf{k}_d} = (E_p + W_B)\Psi^{ex}_{\mathbf{k}_d}$$

[$\hat{\mathscr{H}}_A(R)$ is the Hamiltonian of nucleus A, $E_p + W_B = E_d - \epsilon_d + W_A$] by $\Psi^*_B(R, \mathbf{r}_n)$ and integrate over R and $\mathbf{r}_n$. This gives:

$$\Big[-\frac{\hbar^2}{2M}\nabla^2_{r_p} - E_p\Big]F(\mathbf{r}_p) = -\int \Psi^*_B[V_p + V_{np}]\Psi^{(ex)}_{\mathbf{k}_d}\,dR\,d\mathbf{r}_n, \quad \text{(a)}$$

where the function

$$F(\mathbf{r}_p) = \int \Psi^*_B(R, \mathbf{r}_n)\Psi^{(ex)}_{\mathbf{k}_d}(R, \mathbf{r}_n, \mathbf{r}_p)\,dR\,d\mathbf{r}_n$$

for $r_p \to \infty$ has the form $F(\mathbf{r}_p) \sim r_p^{-1} f_{ex} \exp\{i(k_p r_p - \alpha_p \ln 2k_p r_p)\}$ [the outgoing wave is distorted by the Coulomb field, since the potential on the right side of (a) drops like $1/r_p$ as $r_p \to \infty$]. We consider an equation of somewhat more general form than (a), which we get if we add to both sides of (a) the term $V^0_p(r_p)F(\mathbf{r}_p) = \int \Psi^*_B V^0_p(r_p)\Psi^{(ex)}_{\mathbf{k}_d} \times dR\,d\mathbf{r}_n$, where $V^0_p(r_p)$ is an arbitrary function which goes over into Ze^2/r_p for $r_p \to \infty$:

$$\Big[\nabla^2_{r_p} - \frac{2M}{\hbar^2}V^0_p + k^2_p\Big]F(\mathbf{r}_p) = \frac{2M}{\hbar^2}\int \Psi^*_B[V_{np} + V'_p]\Psi^{(ex)}_{\mathbf{k}_d}\,dR\,d\mathbf{r}_n; \quad \text{(b)}$$

Here $V'_p = V_p - V^0_p$; as $r_p \to \infty$, V'_p drops faster than $1/r_p$.

To solve Eq. (b), we note that the function

$$G(\mathbf{r}_p, \mathbf{r}'_p) = k_p \sum_{l_p m_p} \mathscr{H}_{l_p}(r_p)\,Y_{l_p m_p}(\vartheta_p, \varphi_p) \times L_{l_p}(r'_p)\,Y^*_{l_p m_p}(\vartheta'_p, \varphi'_p), \quad r_p > r'_p$$

(with a similar definition for $r_p < r'_p$, with the coordinates r_p and r'_p interchanged on the right side), satisfies the equation

$$[\nabla^2_{r_p} - 2M\hbar^{-2}V^0_p + k^2_p]\,G(\mathbf{r}_p, \mathbf{r}'_p) = \delta(\mathbf{r}_p - \mathbf{r}'_p)$$

and, for $r_p \to \infty$, has the form

$$G(\mathbf{r}_p, \mathbf{r}'_p) \approx \frac{1}{r_p}\exp\{i(k_p r_p - \alpha_p \ln 2k_p r_p)\} \times \sum_{l_p m_p} \exp\Big\{-i\frac{l_p\pi}{2} + i\eta_{l_p}\Big\} L_{l_p}(r'_p) \times Y^*_{l_p m_p}(\vartheta'_p, \varphi'_p)\,Y_{l_p m_p}(\vartheta_p, \varphi_p) = (1/4\pi r_p)\exp\{ik_p r_p - \alpha_p \ln 2k_p r_p)\}\,\psi^{(-)*}_{\mathbf{k}_p}(\mathbf{r}'_p). \quad \text{(c)}$$

Here $L_{l_p}(r_p)Y_{l_p m_p}(\vartheta_p \varphi_p)$ is the solution, which remains finite for $r_p \to 0$, of the equation

$$[\nabla^2_{r_p} - 2M\hbar^{-2}V^0_p + k^2_p]\,L_{l_p}(r_p)\,Y_{l_p m_p}(\vartheta_p, \varphi_p) = 0, \quad \text{(d)}$$

so that

$$L_{l_p}(r_p) \approx (k_p r_p)^{-1} \sin\left(k_p r_p - \frac{l_p \pi}{2} + \eta_{l_p} - \alpha_p \ln 2k_p r_p\right), \quad \text{if} \quad r_p \to \infty;$$

$\mathcal{H}_{l_p}(r_p) Y_{l_p m_p}(\vartheta_p, \varphi_p)$ is the other solution (singular for $r_p \to 0$) of the same equation, so

$$\mathcal{H}_{l_p}(r_p) \approx (k_p r_p)^{-1} \exp\left\{i\left(k_p r_p - \frac{l_p \pi}{2} + \eta_{l_p} - \alpha_p \ln 2k_p r_p\right)\right\}, \quad \text{if} \quad r_p \to \infty.$$

Finally,

$$\psi_{\mathbf{k}_p}^{(-)}(\mathbf{r}'_p) = 4\pi \sum_{l_p, m_p} \exp\left\{\frac{il_p\pi}{2} - i\eta_{l_p}\right\} L_{l_p}(r'_p)\, Y_{l_p m_p}(\vartheta'_p, \varphi'_p) Y_{l_p m_p}(\vartheta_p, \varphi_p)$$

is the solution of the same equation (d), which for $r_p \to \infty$ becomes a sum of a plane wave (with wave vector $\mathbf{k}_p = k_p \mathbf{r}_p / r_p$) and an outgoing spherical wave distorted by the Coulomb field.

It is clear that the required solution of Eq. (b) has the form:

$$F(\mathbf{r}_p) = -\int G(\mathbf{r}_p, \mathbf{r}'_p)\{2M\hbar^{-2}\int \Psi_B^*(R, \mathbf{r}_n)[V_{np}(|\mathbf{r}_n - \mathbf{r}'_p|) + V'_p(\mathbf{r}'_p, R)]\,\Psi_{\mathbf{k}_d}^{(ex)}(R, \mathbf{r}_n, \mathbf{r}'_p)\,dR\,d\mathbf{r}_n\}\,d\mathbf{r}'_p.$$

For $r_p \to \infty$, $F(r_p)$ according to (c) has the form of an outgoing wave, while (c) gives the value of the amplitude f_{ex} which appears on the right of Eq. (1) in the body of the paper. For the special choice $V_p^0 = Ze^2/r_p$, V'_p has the value $V_p - Ze^2/r_p$ given in the text.

II. CALCULATION OF THE INTEGRAL $I_0(\vartheta_p)$.

(Cf. REFS. 11, 12)

We substitute into the integral (9) given in the text the expressions for the Coulomb functions, where we use for the hypergeometric function $F(\alpha, \gamma, z) = e^{z} F(\gamma - \alpha, \gamma, -z)$ the representation[12]:

$$F(\gamma - \alpha, \gamma, -z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} z^{(1-\gamma)/2} e^{z} \int_0^\infty u^{(\gamma-1-2\alpha)/2} J_{\gamma-1}(2\sqrt{zu})\, e^{-u}\, du. \tag{e}$$

This gives:

$$I_0(\vartheta_p) = \frac{\sqrt{\pi\alpha_p\alpha_d}\exp\{-\pi(\alpha_p + \alpha_d)\}}{\Gamma(1 + i\alpha_p)\,\Gamma(1 + i\alpha_d)} \int_0^\infty v^{i\alpha_d} e^{-v}\, dv \int_0^\infty u^{i\alpha_p} e^{-u}\, du\, X(u, v), \tag{f}$$

$$X(u, v) = \int \exp\{-[\varkappa_n + i(k_d + k_p)]\, r\}\, J_0(2\sqrt{ik_d\eta v})\, J_0(2\sqrt{ik_p\xi' u})\, \frac{d\mathbf{r}}{r}.$$

Here, $\eta = r - \mathbf{r}\cdot\mathbf{k}_d/k_d$, $\xi = r + \mathbf{r}\cdot\mathbf{k}_d/k_d$ (or $\eta' = r - \mathbf{r}\cdot\mathbf{k}_p/k_p$, $\xi' = r + \mathbf{r}\cdot\mathbf{k}_p/k_p$) are parabolic coordinates,

$$\xi' = \xi\cos^2(\vartheta_p/2) + \eta\sin^2(\vartheta_p/2) + 2\sqrt{\xi\eta}\cos(\vartheta_p/2)\sin(\vartheta_p/2)\cos\varphi,$$

where $\cos\vartheta_p = \mathbf{k}_p\cdot\mathbf{k}_d/k_pk_d$, φ is the angle between the planes of the vectors $\mathbf{k}_d$, $\mathbf{k}_p$ and $\mathbf{k}_d$, $\mathbf{r}$. The calculation of $X(u, v)$ is conveniently done in the parabolic coordinates ξ, η, φ, in which $r = \frac{1}{2}(\xi+\eta)$, $d\mathbf{r} = \pi r d\xi d\eta d\varphi/2\pi - 2(ik_pu\xi')^{1/2} = (\sigma^2+\rho^2-2\sigma\rho\cos(\pi-\varphi))^{1/2}$, where $\sigma = 2(ik_p\xi u)^{1/2}\cos(\vartheta_p/2)$, $\rho = 2(ik_pu\eta)^{1/2}\times\sin(\vartheta_p/2)$. According to the addition theorem:

$$J_0(2\sqrt{ik_pu\xi'}) = \sum_{n=-\infty}^{\infty}{}' J_n(\sigma)J_n(\rho)e^{in(\pi-\varphi)},$$

so that the integration over φ in $X(u, v)$ gives simply

$$\int_0^{2\pi} J_0(2\sqrt{ik_pu\xi'})\,d\varphi/2\pi = J_0(\sigma)J_0(\rho),$$

from which $X(u, v) = \pi\mathfrak{J}_1\mathfrak{J}_2$,

where

$$\mathfrak{J}_1 = \int_0^\infty e^{-a\xi}J_0(\sigma)\,d\xi = (1/a)\exp\left\{-i\frac{k_pu}{a}\cos^2\frac{\vartheta_p}{2}\right\};$$

$$\mathfrak{J}_2 = \int_0^\infty e^{-a\eta}J_0(\rho)J_0(2\sqrt{ik_dv\eta})\,d\eta = \frac{1}{2}\exp\left\{-i\frac{k_pu}{a}\sin^2\frac{\vartheta_p}{2} - i\frac{k_dv}{a}\right\}\times J_0\left(\frac{2\sqrt{k_pk_duv}}{a}\sin\frac{\vartheta_p}{2}\right),$$

and $a = \frac{1}{2}[\kappa_n - i(k_p+k_d)]$. In this integration we have used the general formula:

$$\int_0^\infty e^{-au}J_m(\alpha u^{1/2})J_m(\beta u^{1/2})\,du = \frac{i^m}{a}e^{(\alpha^2+\beta^2)/4a}J_m\left(\frac{\alpha\beta}{2ia}\right).$$

Thus

$$X(u, v) = \frac{\pi}{a^2}\exp\{-i(k_pu+k_dv)/a\}\times J_0\left(\frac{2}{a}\sqrt{k_pk_duv}\sin\frac{\vartheta_p}{2}\right).$$

Substituting this value in (f), keeping in mind Eq. (e), and noting that

$$F(\alpha\beta\gamma z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-u}u^{\alpha-1}F(\beta,\gamma,uz)\,du,$$

we get:

$$I_0(\vartheta_p) = \frac{\pi^{3/2}\sqrt{\alpha_p\alpha_d}}{a^2}\left(\frac{a}{a+ik_d}\right)^{i\alpha_d+1}\times\left(\frac{a}{a+ik_p}\right)^{i\alpha_p+1}e^{-\pi(\alpha_p+\alpha_d)}\times F(i\alpha_d+1, i\alpha_p+1, 1, -\zeta),$$

where $\zeta = \frac{k_dk_p}{(a+ik_d)(a+ik_p)}\sin^2\frac{\vartheta_p}{2} = \zeta_0\sin^2\frac{\vartheta_p}{2}$.

Substituting the value

$$F(i\alpha_d+1, i\alpha_p+1, 1, -\zeta) = (1+\zeta)^{-i\alpha_p-i\alpha_d-1}F(-i\alpha_d, -i\alpha_p, 1, -\zeta),$$

we then have:

$$I_0(\vartheta_p) = \frac{\pi^{3/2}\sqrt{\alpha_p\alpha_d}\exp\{-\pi(\alpha_d+\alpha_p)\}}{(a+ik_p)(a+ik_d)}\times\left(\frac{a}{(a+ik_d)(1+\zeta)}\right)^{i\alpha_d}\times\left(\frac{a}{(a+ik_p)(1+\zeta)}\right)^{i\alpha_p}\frac{F(-i\alpha_d, -i\alpha_p 1, -\zeta)}{1+\zeta}.$$

After some elementary transformations, one gets from this to the value given in the text.

III. THE ASYMPTOTIC FORM OF $|F(i\alpha_d, i\alpha_p, 1, -\zeta)|^2$ FOR $\alpha_d > 1$

For $\alpha_d > 1$, the contour integral

$$F(i\alpha_d, i\alpha_p, 1, -\zeta) = \frac{1}{2\pi i}\int_C \left(\frac{u}{u-1}\right)^{i\alpha_p}(1+\zeta u)^{-i\alpha_d}\frac{du}{u} = \int_C \exp\{\alpha_d\varphi(u)\}\frac{du}{2\pi i u}$$

(where the contour C encircles the points + 1 and − 1 on the real axis) can be evaluated by the saddle-point method; the result, according to Sommerfeld[11], is

$$|F(i\alpha_d, i\alpha_p, 1, -\zeta)|^2 = |\exp\{\alpha_d\varphi(u_0)\}(2\pi\alpha_d|\varphi_0''|u_0)^{-1/2}|^2, \quad \text{(g)}$$

where

$$\varphi(u) = i\ln u^{\rho}(u-1)^{-\rho}(1+\zeta u)^{-1}, \quad \rho = \alpha_p/\alpha_d;$$

$$u_0 = \frac{1-\rho}{2} + \frac{i}{2}\sqrt{\frac{4\rho}{\zeta} - (1-\rho)^2}$$

is the solution of the equation $\varphi'(u_0) = 0$,

$$|\varphi_0''| = |\varphi''(u_0)| = \left|-\zeta\frac{2u_0-(1-\rho)}{(u_0^2-u_0)\left(\frac{1}{\zeta}+u_0\right)}\right| = \frac{\zeta^2\sqrt{(4\rho/\zeta)-(1-\rho)^2}}{\rho(1+\zeta)}$$

[in the last equation we use the fact that $|(u^2 - u_0)(u_0 + \zeta^{-1})| = |-\rho(u_0 + \zeta^{-1})^2| = \rho(1+\zeta)\zeta^{-2}$]. It is assumed throughout that $\rho < 1$, but since $F(i\alpha_d, i\alpha_p, 1, -\zeta)$ is symmetric in α_d and α_p, the final formula is also valid for $\rho > 1$. Substituting the values of $|\varphi_0''|$ and u_0 in (g), and using $|u_0|^2 = \rho/\zeta$ and

$$\alpha_d[\varphi(u_0) + \varphi^*(u_0)] = 2\pi\alpha_p + 2(\alpha_d\psi_d - \alpha_p\psi_p),$$

where ψ_d and ψ_p are the angles defined in (16), we obtain for $|F(i\alpha_d, i\alpha_p, 1, -\zeta)|^2$ the value (15) given in the text. The saddle-point method is applicable if

$$\alpha_d|\varphi_0''| = \zeta^2[(\alpha_d\alpha_p/\zeta) - (\alpha_p-\alpha_d)^2]^{1/2}/\rho(1+\zeta) > 1.$$

This condition limits the values of ζ to those satisfying the inequality $\zeta > \zeta'$ given in the text.

Translated by M. Hamermesh
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