Real Spinors in Curvilinear Coordinates and Pseudo-Riemannian Space

G. A. ZAITSEV
Ivanova, USSR
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Properties of matrix-tensors are utilized for the investigation of real spinors in curvilinear coordinates. It is shown that one need not replace ordinary derivatives of spinors by more general derivatives. In connection with this, contrary to the opinion that is widely held, the equations of Dirac and other analogous relativistically invariant differential equations, by nature, do not change their form in transformation into curvilinear coordinates. The problem of generalization to the case of pseudo-Riemannian space is considered. It is significant that one can explain the appearance of a term with the factor $m_0 c/\hbar$ in the equation of Dirac, if we regard the space, not as pseudo-Euclidian, but as pseudo-Riemannian.

1. INTRODUCTION

IN earlier publications[1-3], it is shown that real spinors, applied to the description of the state of individual elementary particles, must be examined as parameters defined by certain primary tensors. Thus, in order for it to be possible to write down appropriate equations in curvilinear coordinates, and in order to discover ways of possible generalization, it is necessary to examine the question of real spinors in curvilinear coordinates and pseudo-Riemannian space. The method by which it is possible to do this is based on the law of transformation of a component tensor under change from one coordinate system to another, provided that the appropriate related matrix is employed. It is necessary to go into some detail on this point, if only for the reason that the results obtained in this case are essentially different from the results of a series of authors who have attempted to generalize the equations of Dirac[4-10]

The question of writing the equations for elementary particles, containing real spinors, in curvilinear coordinates and consequent generalizations in the case of pseudo-Riemannian space should present considerable physical interest. The general theory of relativity showed that mass, energy and momentum appear to be related to the properties of space. Various investigators[11] hold that the general theory of relativity is, above all, a theory of gravitation. It is impossible to agree with such an opinion. The fundamental meaning of the general theory of relativity consists of the following: it permits us to find important properties of energy (and mass) and momentum, and to display the close connection between these and the properties of four-dimensional space-time. We should consider the theory of gravitation and certain confirmations of the general theory of relativity, of an experimental character, as confirmation of the correctness of the geometric treatment of important physical quantities--components of the energy tensor--momentum. As for the theory of elementary particles, it appears characteristic that the question of the more profound nature of mass, energy and momentum remains unclear. Thus, for instance, in the equation for the electron the rest mass m_0 is introduced in a purely formal way, but its real meaning is obscure, etc. Inasmuch as the geometric nature of quantities of such a type is confirmed for macroscopic phenomena, then, in their consideration in the theory of microphenomena, it is necessary to take into account the possibility of their connection with the geometric characteristics of space. This question must be examined carefully in each case, in any re-examination of the basic foundations of the theory of microphenomena.

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[1] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 653 (1953)

[2] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 667 (1953)

[3] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 29, 166 (1955); Soviet Phys. 2, 240 (1956)

[4] V. A. Fock, Z. Physik 57, 261 (1929); Zh. Ros. Fiz.-Khim. Ob., Fiz. Ch. 62, 133 (1930)

[5] H. Weyl, Z. Physik 56, 330 (1929)

[6] E. Schrodinger, Sitzungsb. Preus. Akad. 105 (1932)

[7] V. Bargmann, Sitzungsb. Preus. Akad. 346 (1932)

[8] L. Infeld and B. L. van der Waerden, Sitzungsb. Preus. Akad. 380 (1933)

[9] J. A. Schouten, J. Math. Phys. 10, 239 (1931)

[10] E. Cartan, *Leçons sur la theorie des spineurs*, Paris, Hermann and Cie., 1938

[11] V. A. Fock, J. Exper. Theoret. Phys. USSR 9, 375 (1939)

2. MATRIX-TENSORS IN CURVILINEAR COORDINATES

Let us examine some properties of matrix-tensors in curvilinear coordinates. Just because of a lack of knowledge of these properties various misunderstandings may arise in the discussion of questions concerning real spinors.

For definiteness, we shall distinguish quantities, expressed in orthogonal coordinates and investigated in previous discussions[2], that will carry on the right hand the index zero, for instance, $g^0_{\alpha\beta}$, x_0^α, R^0_α, J_0, etc. To transform to curvilinear coordinates $x^\alpha = x^\alpha(x_0^1, x_0^2, x_0^3, x_0^4)$ it is necessary to replace R^0_α by R_α according to the formula

$$R_\alpha = \frac{\partial x_0^\beta}{\partial x^\alpha} R^0_\beta, \tag{1}$$

in such a way that $R_\alpha = \partial X / \partial x^\alpha$, where $X = x_0^\beta R^0_\beta = x^\beta R_\beta$ is a matrix- vector. Further, we have

$$\begin{aligned} {}^1/_2 (R_\alpha R_\beta + R_\beta R_\alpha) &= g_{\alpha\beta} \\ &= \frac{\partial x_0^\gamma}{\partial x^\alpha}\frac{\partial x_0^\delta}{\partial x^\beta} g^0_{\gamma\delta}, \; R^\gamma = g^{\gamma\alpha} R_\alpha \end{aligned} \tag{2}$$

$${}^1/_2 (R_\beta R^\gamma + R^\gamma R_\beta) = \delta^\gamma_\beta,$$

$${}^1/_2 (R^\alpha R^\beta + R^\beta R^\alpha) = g^{\alpha\beta}.$$

From Eqs. (1) and (2) we see that matrices R_α and R^α, under transformation from one system of coordinates to another, transform formally as components of a covariant and contravariant vector. As far as matrix-tensors are concerned, in general they do not vary under these transformations, for instance,

$$F = {}^1/_2 F^0_{\alpha\beta} R_0^\alpha R_0^\beta = {}^1/_2 F_{\alpha\beta} R^\alpha R^\beta \tag{3}$$

etc.

Consider further in what form the matrix J must be defined in curvilinear coordinates. It is expedient to write the formula for J_0 in the form

$$J_0 = R_0^1 R_0^2 R_0^3 R_0^4 = -\frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} R_0^\alpha R_0^\beta R_0^\gamma R_0^\delta. \tag{4}$$

The minus sign is related to the fact that we assume $\epsilon^{1234} = 1$, while $\epsilon_{1234} = -1$. It is then clearly necessary to define J in the general case as

$$J = -\frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} R^\alpha R^\beta R^\gamma R^\delta. \tag{5}$$

Further, making use of the fact that

$$g = |g_{\alpha\beta}| = \left| \frac{\partial x_0^\gamma}{\partial x^\alpha}\frac{\partial x_0^\delta}{\partial x^\beta} g^0_{\gamma\delta} \right| = \left| \frac{\partial x_0^\beta}{\partial x^\alpha} \right|^2 g_0, \tag{6}$$

$$g_0 = -1,$$

and likewise that

$$\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x_0^\alpha}{\partial x^{\alpha'}}\frac{\partial x_0^\beta}{\partial x^{\beta'}}\frac{\partial x_0^\gamma}{\partial x^{\gamma'}}\frac{\partial x_0^\delta}{\partial x^{\delta'}} = \left| \frac{\partial x_0^\mu}{\partial x^\nu} \right| \varepsilon_{\alpha'\beta'\gamma'\delta'}$$

[see reference 12, Eq. (14)] and

$$R_0^\alpha = \frac{\partial x_0^\alpha}{\partial x^\beta} R^\beta \quad \left(R^\beta = \frac{\partial x^\beta}{\partial x_0^\gamma} R_0^\gamma \right), \tag{7}$$

from Eqs. (4) and (5) we obtain

$$J_0 = \sqrt{-g} J, \; J = (1/\sqrt{-g}) J_0. \tag{8}$$

And, as in reference 2, it is necessary to obtain the formulas for the result of multiplying J by other matrices. For instance,

$$R_\alpha J = -J R_\alpha = -\frac{1}{3!} \varepsilon_{\alpha\beta\gamma\delta} R^\beta R^\gamma R^\delta \tag{9}$$

etc. As a demonstration it suffices to transform to orthogonal coordinates and make use of the corresponding formulas from Veblen[12].

Taking into account the fact that R^α or R_α appear as linear combinations of the four constant matrices R_0^α, and considering that explicit expressions for matrices are defined according to Veblen, we obtain

$$R'_\alpha R_0 = R_0 R_\alpha, \; (R^\alpha)' R_0 = R_0 R^\alpha, \tag{10}$$

$$J' R_0 = R_0 J \qquad (R_0 = R_0^1 R_0^2 R_0^3),$$

whence the matrix R_0 has one and the same value at any point of space.

In conformity with the well-known formulas for differentiation of fundamental vectors in curvilinear coordinates of pseudo-Euclidian space (see reference 13, Sec. 77) we will have

$$dR_\alpha = \Gamma^\gamma_{\alpha\beta} R_\gamma dx^\beta, \quad \partial R_\alpha / \partial x^\beta = \Gamma^\gamma_{\alpha\beta} R_\gamma, \tag{11}$$

where

$$\Gamma^\gamma_{\alpha\beta} = {}^1/_2 g^{\gamma\delta} [(\partial g_{\delta\alpha} / \partial x^\beta) + (\partial g_{\delta\beta} / \partial x^\alpha) - (\partial g_{\alpha\beta} / \partial x^\delta)].$$

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[12] O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge Univ. Press, 1927

[13] P. K. Rashevskii, *Riemannian Geometry and Tensor Analysis*, State Publishing House of Technical-Theoretical Literature, 1953

From Eq. (11) it follows also that

$$dR^\alpha = -\Gamma^\alpha_{\gamma\beta} R^\gamma dx^\beta, \quad \partial R^\alpha / \partial x^\beta = -\Gamma^\alpha_{\gamma\beta} R^\gamma. \quad (12)$$

Equations (11) and (12) permit us to find differentials and derivatives of matrix-tensors, with components expressed in curvilinear coordinates. In this case there occurs an important property of such differentials and derivatives which we consider in the example of a matrix-tensor of the second rank $F = \frac{1}{2} F^{\alpha\beta} R_\alpha R_\beta$. According to Eq. (11) we have

$$dF = \tfrac{1}{2}\left(\frac{\partial F^{\alpha\beta}}{\partial x^\delta} R_\alpha R_\beta + F^{\alpha\beta}\Gamma^\gamma_{\alpha\delta} R_\gamma R_\beta + F^{\alpha\beta} R_\alpha \Gamma^\gamma_{\beta\delta} R_\gamma\right) dx^\delta = \tfrac{1}{2}(DF^{\alpha\beta}) R_\alpha R_\beta, \quad (13)$$

where

$$DF^{\alpha\beta} = [(\partial F^{\alpha\beta}/\partial x^\gamma) + \Gamma^\alpha_{\delta\gamma} F^{\delta\beta} + \Gamma^\beta_{\delta\gamma} F^{\alpha\delta}]\, dx^\gamma. \quad (14)$$

We see that $DF^{\alpha\beta}$ is the absolute differential of $F^{\alpha\beta}$. Hence, it follows that the ordinary derivative of a matrix-tensor is expressed by covariant derivatives of its components according to formulas of the type

$$\partial F / \partial x^\gamma = \tfrac{1}{2} (F^{\alpha\beta})_\gamma R_\alpha R_\beta. \quad (15)$$

In obtaining our formulas, the concrete form of the matrix R_α was not defined. It is only important to emphasize that, for convenience in the choice of matrices, R^0_α must be constant. This corresponds to that fact that, for a given constant vector at any point of space there must correspond the very same matrix, i.e., the law of correspondence between matrices and tensors must be one and the same throughout all space*.

Finally, making use of Eq. (8), we find the expression for the differentiation of the matrix J

$$\frac{\partial J}{\partial x^\alpha} = -\frac{1}{2}\frac{\partial \ln g}{\partial x^\alpha} J = -\Gamma^\beta_{\beta\alpha} J. \quad (16)$$

* If we define the law of correspondence in various forms for different points of space, then a set of completely indeterminate conditions would enter. Thus, speaking no longer of the fact that it is necessary to take this into account in the differentiation of matrices, in ordinary cases we should not obtain either the results of reference 2 or of other works , since the fundamental matrices were considered constant here.

3. REAL SPINORS IN CURVILINEAR COORDINATES

In reference 2, the components of a real spinor ψ were defined as parameters which were characterized by the assignment of an anti-symmetric tensor of the second rank, for which both invariants equal zero. Then ψ is found from

$$F_0^{\alpha\beta} = -\psi' R_0 R_0^\alpha R_0^\beta \psi. \quad (17)$$

Real spinors can also be defined by starting from the assignment of other primary tensors (see reference 3)*. For definiteness, we restrict ourselves to Eq. (17) and by its help take up the question of real spinors in curvilinear coordinates, although we would obtain other results if we were to examine, for instance, systems of other real spinors defined by certain tensors (as in reference 3). Transforming to curvilinear coordinates, in accordance with Eq. (3) we obtain

$$F^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x_0^\gamma}\frac{\partial x^\beta}{\partial x_0^\delta}(-\psi' R_0 R_0^\gamma R_0^\delta \psi) = -\psi' R_0 R^\alpha R^\beta \psi. \quad (18)$$

We can also employ this formula for the definition of ψ in generalized coordinates.

In the transformation to curvilinear coordinates, the matrices R_0^α in expressions for components of tensors are replaced by R^α, but the corresponding real spinors remain invariant. Also, the matrix R_0 does not change**.

* We note that in the literature the term "real spinors" is already used, although their treatment differs from ours. That is, matrices R_0^α, $R_0^\alpha R_0^\beta (\alpha \neq \beta)$, etc. form the idea of hyper-complex systems of numbers (sedenions). The space in which this representation is realized, is often called spinor, but the tensors of this space, according to van der Waerden, are also called spinors (or also wave tensors). If the matrices R_0^α have real elements, then spinors with real elements are called real spinors (see reference 14, J. A. Schouten and D. v. Dantzig, Z. Physik **78**, 639 (1932), p. 657]. For our definition of real spinors, as parameters characterizing primary tensors, some of the properties coincide with the attributes of spinors, as defined above. But there are differences connected with the fact that a perfectly definite meaning is attached to our real spinors. Thus, components of a real spinor are transformed only in rotations or reflections of all space, but not in changes of the choice of the system of coordinates. In a symmetric transformation the general multiplier i can appear; otherwise, the problem arises as to transformation to a description in curvilinear coordinates, etc.

** Moreover, these statements also apply in the case of double real spinors, defined in correspondence to the assignment of certain tensors.

[14] J. A. Schouten and D. v. Dantzig, Z. Physik **78**, 639 (1932)

The problem of differential operations does not present complications, since expressions for the transformation of matrices R^{α} (or R_{α}) are known Making use of Eq. (11), for instance, we obtain the ordinary differential of $F^{\alpha\beta}$:

$$dF^{\alpha\beta} = (\partial F^{\alpha\beta} / \partial x^{\gamma})\, dx^{\gamma}$$

$$= [- (\partial\psi' / \partial x^{\gamma})\, R_0 R^{\alpha} R^{\beta}\psi$$

$$- \psi' R_0 R^{\alpha} R^{\beta} (\partial\psi / \partial x^{\gamma})$$

$$+ \Gamma^{\alpha}_{\delta\gamma}\psi' R_0 R^{\delta} R^{\beta}\psi + \Gamma^{\beta}_{\delta\gamma}\psi' R_0 R^{\alpha} R^{\delta}\psi]\, dx^{\gamma},$$

i.e.,

$$DF^{\alpha\beta} = - (d\psi')\, R_0 R^{\alpha} R^{\beta}\psi - \psi' R_0 R^{\alpha} R^{\beta}\, d\psi, \tag{19}$$

where the quantity on the left is the absolute differential of $F^{\alpha\beta}$. If we introduce the concept of "covariant derivative" of real spinors $(\psi)_{\gamma}$ in conformity with the formula

$$(F^{\alpha\beta})_{\gamma} = - (\psi)'_{\gamma} R_0 R^{\alpha} R^{\beta}\psi - \psi' R_0 R^{\alpha} R^{\beta} (\psi)_{\gamma}, \tag{20}$$

where $(F^{\alpha\beta})_{\gamma}$ are ordinary covariant derivatives of tensors, then we get from Eq. (19)

$$(\psi)_{\gamma} = \partial\psi / \partial x^{\gamma}, \tag{21}$$

so that "covariant derivatives" of real spinors coincide with ordinary derivatives. Hence, there follows, in particular, an important consequence, that $R^{\alpha}_{0}\,(\partial\psi / \partial x^{\alpha}_{0})$, for transformation into curvilinear coordinates, is replaced by $R^{\alpha}(\partial\psi/\partial x^{\alpha})$, i.e., in spite of long-standing notions, it has in essence the same form (only now the matrices R^{α} are no longer constant). Thus, for instance, Maxwell's equations for the radiation field with a fixed direction of propagation, written in the form of reference 15, have, in curvilinear coordinates, the form of

$$R^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} = 0 \left(\psi' R_0 \frac{\partial\psi}{\partial x^{\alpha}} = 0, \quad \psi' R_0^4 \frac{\partial\psi}{\partial x^{\alpha}} = 0\right). \tag{22}$$

From Eq. (22) one may conversely obtain the corresponding Maxwell equations in the customary form but in curvilinear coordinates. Thus, multiplying Eq. (20) on the left by $\psi' R_0\, R^{\beta}$, and the transposed equation on the right by $R_0\, R^{\beta}\psi$, consolidating the resulting expressions, and making use of Eqs. (2), (10) and (20), there results

$$(F^{\beta\alpha})_{\alpha} = 0$$

etc. Likewise it would have been possible to write the equations of Dirac (expressed as equations for two real spinors, see reference 16) in curvilinear coordinates, etc.

Note that the equation of Dirac , written in the form indicated above, will have a different appearance from the Dirac equations in curvilinear coordinates, obtained by the method of other authors, inasmuch as more general expressions are used in them in place of the derivatives $\partial\psi/\partial x^{\alpha}$. Therefore, we shall consider this problem in some detail.

It is a characteristic feature of researches concerned with the equations of Dirac in curvilinear coordinates and pseudo-Riemannian space that (if one looks at spinors from our point of view) there are various ideas mixed together: the transformation from one system of coordinates to another and the transformation from one isomorphic correspondence between matrices and tensors to others. As an example of this kind, consider the work of Fock [4], since the views set forth there have received wide distribution and have had significant influence on other work*. Fock introduces at every point of four-dimensional space an orthogonal reference system and refers the components of the four-dimensional vector current to this reference system. Components of vectors in such a local system of coordinates are expressed by components of spinors and constant matrices, the same for all reference systems. Making use of our notation, this means that the matrices R^{0}_{α}, corresponding to the basic vectors of reference system, are considered constant at every point of space. But since the vasic vectors of the reference system, under transformation from one point to another, are changed, then in the use of curvilinear coordinates in pseudo-euclidean space, the matrices R^{0}_{α} are no longer the same at any point, and are linked to one another by an appropriate transformation. In other words, different matrices correspond at each point to the same constant vector of pseudo-

[15] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 675 (1953)

* Detailed accounts of these views are contained in reference 17.

[16] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 29, 176 (1955); Soviet Phys. 2, 140 (1956)

[17] A. P. Sokolov and D. D. Ivanenko, *Quantum field theory*, State Publishing House of Technical- Theoretical Literature, 1952

euclidian space, whence the law of correspondence that depends on what form is chosen for the local reference system. For just this reason, our way of writing the equation of Dirac in curvilinear coordinates differs from the writings of Fock.

Similar considerations apply also to other researches to which we have referred.

4. TRANSFORMATION IN PSEUDO-RIEMANNIAN SPACE

Separate formulas, written for curvilinear coordinates, are correct in the more general case of pseudo-Riemannian space. Here also for every point of space it is necessary to introduce matrices R^α or R_α, with the help of which the expressions for matrix-tensors are written out. Eqs. (1) to (9) apply here as before for the corresponding matrices. But it is necessary to keep in mind that, insofar as the space is no longer pseudo-euclidian, the matrices R_0^α, R_0, J_0 will now characterize some locally-Galilean systems of coordinates, whereupon they can no longer be considered constant for all space (if only not to confuse a transformation from one system of coordinates to another, and the different character of an isomorphic correspondence between matrices and tensors). So far as the operation of differentiation is concerned, the situation is considerably more complicated. If we write

$$\partial' R^\alpha / \partial x^\beta_s = -\Gamma^\alpha_{\gamma\beta} R^\gamma, \tag{23}$$

then the right side will not be an absolute derivative of the matrices R^α) (A prime is written to denote this case.) In fact, from Eq. (23) we obtain

$$\frac{\partial'}{\partial x^\gamma}\left(\frac{\partial' R^\alpha}{\partial x^\beta}\right) - \frac{\partial'}{\partial x^\beta}\left(\frac{\partial' R^\alpha}{\partial x^\gamma}\right) = B^\alpha_{\delta\gamma\beta} R^\delta, \tag{24}$$

where $B^\alpha_{\delta\gamma\beta} = \partial\Gamma^\alpha_{\delta\gamma}/\partial x^\beta - \partial\Gamma^\alpha_{\delta\beta}/\partial x^\gamma + \Gamma^\epsilon_{\delta\gamma}\Gamma^\alpha_{\epsilon\beta} - \Gamma^\epsilon_{\delta\beta}\Gamma^\alpha_{\epsilon\gamma}$ are components of the curvature tensor*. Therefore, $\partial' R^\alpha / \partial x^\beta$ can be derived only if $B^\alpha_{\delta\gamma\beta} = 0$, i.e., if the space is pseudo-euclidean. For absolute derivatives we must write the expression

$$\partial R^\alpha / \partial x^\beta = -\Gamma^\alpha_{\gamma\beta} R^\gamma + \overline{Y}^\alpha_\beta, \tag{25}$$

where matrices Y^α_β characterize the departure of the space from pseudo-euclidian and for the latter must revert to zero.

* Various authors denote components of the full curvature tensor differently. Our notation corresponds to that of reference 12.

Real spinors in pseudo-Riemannian space must be defined as in curvilinear coordinates, i.e., from relations of the type $F^{\alpha\beta} = -\psi' R_0 R^\alpha R^\beta \psi$, etc., but the matrix R_0 will now, generally speaking, be different for different points, and the question of differentiation and of differential equations, containing real spinors, becomes considerably complicated. Therefore, we limit ourselves to the selection of a special case, when the matrices R^α at any point of space are linear combinations of the five *constant* matrices $R^\alpha_{(0)}$ and $J_{(0)}$. (We assume $R^\alpha_{(0)}$ and $J_{(0)}$ are such that Eq. (10) is satisfied.) In this case the matrix R_0 will also be constant. The meaning of the limitation that we have placed on the matrices R^α is that pseudo-Riemannian space is considered as a certain surface in five-dimensional pseudo-Euclidean space, for which there are constant basic matrices —either $R^\alpha_{(0)}$ or $J_{(0)}$ (i.e., it will be a space of the first class in space, see reference 18). In connection with this, the derivatives of R^α will be linear combinations of the matrices $R^\alpha_{(0)}$ and $J_{(0)}$ or R^β and J,

$$\partial R^\alpha / \partial x^\beta = -\Gamma^\alpha_{\gamma\beta} R^\gamma + \pi^\alpha_\beta J, \tag{26}$$

i.e., where the nonvanishing character of π^α_β is connected with the non-Euclidean nature of the space.

It is not difficult to show that if Eq. (26) is satisfied,

$$[(R^\alpha + dR^\alpha)(R^\beta + dR^\beta) \tag{27}$$

$$+ (R^\beta + dR^\beta)(R^\alpha + dR^\alpha)] - [R^\alpha R^\beta + R^\beta R^\alpha]$$

$$= 2\frac{\partial g^{\alpha\beta}}{\partial x^\gamma} dx^\gamma, \quad dR^\alpha = \frac{\partial R^\alpha}{\partial x^\gamma} dx^\gamma,$$

The deviation of the space from pseudo-Euclidean will be shown by the nature of the differential equations that contain real spinors. It is necessary to examine how equations for primary tensors, which are written, however, in original form, and properties of space will influence the nature of transformation from equations for tensors into equations for spinors.

Let us consider the case where the primary tensors are vectors and pseudovectors with components $P^\alpha_{(+)}$ and n^α and the invariants and pseudo-invariants Ω_1 and Ω_2, connected with the cor-

[18] L. P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press, 1926

responding relations (see references 3 and 16).
n^α is expressed by the real spinors ψ_1 and ψ_2 in the form

$$n^\alpha = \sqrt{-g}\,\psi'_{(2)} R_0 R^\alpha \psi_{(1)}. \qquad (28)^*$$

For abbreviation we will make use of the components $Q^\alpha = (1/\sqrt{-g})\,n^\alpha$ in place of the components vector of the pseudovector.

Suppose that we have the relativistically invariant differential equation

$$(Q^\alpha)_\alpha = 0. \qquad (29)$$

If the space be pseudo-Euclidean, then Eq. (29) would be obtained, in particular, from the following relativistically invariant equations for ψ_1 and ψ_2:

$$R^\alpha \frac{\partial}{\partial x^\alpha} \psi_{(1)} = 0; \qquad (30)$$

$$R^\alpha \frac{\partial}{\partial x^\alpha} \psi_{(2)} = 0. \qquad (31)$$

For pseudo-Riemannian space Eq. (29) no longer follows from Eqs. (30) and (31).

Supposing that Eq. (26) holds, we consider the problem of the form to which it is necessary to generalize Eqs. (30) and (31), so that as before there will be equivalence with Eq. (29). From Eqs. (26) and (28) results

$$(Q^\alpha)_\alpha = \frac{\partial Q^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\gamma\alpha} Q^\gamma \qquad (32)$$

$$= \frac{\partial \psi'_{(2)}}{\partial x^\alpha} R_0 R^\alpha \psi_{(1)} + \psi'_{(2)} R_0 R^\alpha \frac{\partial \psi_{(1)}}{\partial x^\alpha} + \pi^\alpha_\alpha \psi'_{(2)} R_0 J \psi_{(1)}.$$

Consequently, in order that Eq. (29) hold for the operator it is necessary to add $\frac{1}{2}\pi^\alpha_\alpha J$ to the operator $R^\alpha(\partial/\partial x^\alpha)$ so that Eqs. (30) and (31) become

$$R^\alpha \frac{\partial}{\partial x^\alpha} \psi_{(1)} + {}^1\!/_2 \pi^\alpha_\alpha J \psi_{(1)} = 0; \qquad (33)$$

$$R^\alpha \frac{\partial}{\partial x^\alpha} \psi_{(2)} + {}^1\!/_2 \pi^\alpha_\alpha J \psi_{(2)} = 0. \qquad (34)$$

In fact, to obtain Eq. (29) it is necessary to multiply Eq. (33) on the left by $\psi'_2 R_0$, but the equation transformed into Eq. (34), on the right by $R_0 \psi_{(1)}$, and to add the result.

Equations (33) and (34) will be the equations of Dirac without an external field (see reference 16) if we merely set

$$-{}^1\!/_2 \pi^\alpha_\alpha = m_0 c / \hbar. \qquad (35)$$

Thus, it seems that the appearance of a term proportional to the rest mass in the equations for elementary particles can be connected with the curvature of four-dimensional space. This result presents great interest of a fundamental nature, since it shows how a term characterizing the mass of a particle can appear in the transition from tensor equations in non-Euclidean space into equations containing spinors. Already at that time, possibly, it has only approximate significance, and in the future theory of the internal structure of elementary particles, one must expect essential changes and refinements.

In conclusion, we shall derive a formula relating π^α_β with components of the curvature tensor. From Eqs. (26) and (5), taking account of Eq. (9), we obtain, after some computation,

$$\frac{\partial J}{\partial x^\beta} = -{}^1\!/_2 \frac{\partial \ln g}{\partial x^\beta} J - \frac{\pi_{\alpha\beta}}{g} R^\alpha \qquad (36)$$

$$(\pi_{\alpha\beta} = g_{\alpha\gamma} \pi^\gamma_\beta).$$

Taking into account that $\partial^2 R^\alpha / \partial x^\gamma \partial x^\beta = \partial^2 R^\alpha / \partial x^\beta \partial x^\gamma$, we obtain the formula relating matrices Y^α_β [see Eq. (25)] with components of the curvature tensor

$$\frac{\partial Y^\alpha_\gamma}{\partial x^\beta} - \frac{\partial Y^\alpha_\beta}{\partial x^\gamma} + \Gamma^\alpha_{\delta\beta} Y^\delta_\gamma - \Gamma^\alpha_{\delta\gamma} Y^\delta_\beta = B^\alpha_{\delta\gamma\beta} R^\delta. \quad (37)$$

Hence, using Eqs. (26) and (36) for examination of our case, we obtain the following equations which π^α_β must obey:

$$\frac{\partial \pi^\alpha_\gamma}{\partial x^\beta} + \Gamma^\alpha_{\delta\beta} \pi^\delta_\gamma - \Gamma^\delta_{\gamma\beta} \pi^\alpha_\delta - {}^1\!/_2 \frac{\partial \ln g}{\partial x^\beta} \pi^\alpha_\gamma \qquad (38)$$

$$= \frac{\partial \pi^\alpha_\beta}{\partial x^\gamma} + \Gamma^\alpha_{\delta\gamma} \pi^\delta_\beta - \Gamma^\delta_{\gamma\beta} \pi^\alpha_\delta - {}^1\!/_2 \frac{\partial \ln g}{\partial x^\gamma} \pi^\alpha_\beta;$$

$$\frac{1}{g} (\pi^\alpha_\beta \pi_{\delta\gamma} - \pi^\alpha_\gamma \pi_{\delta\beta}) = B^\alpha_{\delta\gamma\beta} \qquad (39)$$

This is found to be in complete agreement with the theory of space of the first class [see reference 18, p. 238, Eqs. (59.3) and (59.4)].

We see that the π^α_β are not related to the components of the curvature tensor in exactly the same way as the components of tensors of energy-momentum in the corresponding form in the general

* The presence on the right side of Eq. (28) of factor $\sqrt{-g}$, which is equal to unity in the particular case of orthonormal basic vectors, is related to certain requirements for the quantities $\psi_{(1)}$, $\psi_{(2)}$. We examine just this case, therefore, which here is especially easy to show, in what form terms which conserve the rest mass appear in the equations for $\psi_{(1)}$, $\psi_{(2)}$. Although the situation is actually more complicated, the principle of the problem remains the same.

theory of relativity. But, it is necessary to keep in mind that for elementary particles the concrete character of the connection between the mass and the properties of space does not necessarily have to correspond to the particular requirements of the theory, which are confirmed only for macrophenomena.

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Phenomena in the Vicinity of Detonation Formation in a Gas

K. I. SHCHELKIN
Institute of Chemical Physics, Academy of Sciences, USSR
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Phenomena in the vicinity of a detonation are discussed. It is shown in particular that in accordance with a previously developed theory[1,2] explaining how slow burning combustion turns into a detonation, a detonation can occur in gas both at some distance in front of the slow combustion as well as in its immediate vicinity.

INTRODUCTION

THE mechanism whereby slow combustion of a gas in a tube turns into a detonation was discussed earlier[1,2] and can be summarized briefly as follows: The expansion of the slowly-burning mixture causes motion and turbulence of the unburned gases. The turbulence increases the velocity of propagation of the combustion relative to the gas, and this in turn causes an increase in the velocity of the gas--the combustion accelerates progressively.

The accelerating combustion, acting like a piston moving in a gas-filled tube, produces an adiabatic-compression wave. The slope of the adiabatic-compression wave front increases progressively until a state and velocity discontinuity occurs in the gas. At the instant that the discontinuity occurs, its surface separates the undisturbed and uncompressed adiabatically compressed gas, the velocity of which is readily computed from the velocity of sound in the gas on both sides of the discontinuity.

From the theory of random discontinuities[3] it is known that such a discontinuity of state and velocity cannot propagate in the gas; it breaks into a shock wave, which travels through the unperturbed gas, and a rarefaction wave, which propagates in the opposite direction, through the adiabatically-compressed gas. At the place of shock-wave formation the gas experiences a density and temperature discontinuity, the surface of which is stationary relative to the gas. The gas temperature in the shock wave rises sharply because of the non-adiabatic shock compression. This leads to a detonative ignition of the uncombusted gas--to the formation of a detonation.

Once certain assumptions are made, the entire process lends itself readily to analysis; this was done with an accuracy to within constant multipliers in the reference quoted[2]. The fact that in most cases the explosion actually occurs not in a single plane but over a certain length of the tube does not affect the argument substantially. It produces no change whatever in the qualitative picture of the detonation phenomenon and reduces only insignificantly the accuracy of the computation of the distance between the ignition point and the location where the detonation occurs.

In principle it is possible also to suggest another mechanism for the pre-detonation acceleration of the combustion in the tube, proposed by L. D. Landau, and based on the instability of the plane combustion front and the self-turbulence of the gas in the region of the flame. In tubes, however, it is the turbulence produced by the walls that always precedes the self-turbulence and determines the acceleration of the flame. To observe the self-turbulence it becomes necessary to employ special measures to prevent formation of turbulence due to the walls[4].

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