The Problem of the Interpretation of Dirac's Equation for the Electron

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It is shown that Dirac's equation for the electron can be considered a system of equations for two real spinors. Gauge invariance of the equations will correspond to invariance under spinor transformations of the second kind. Consequently, in Dirac theory, it is not the components of ψ which should be considered as basic quantities, but rather the definite tensors which allow one to find the corresponding real spinors to within the spinor transformation.

THE properties of tensors characterized by two real spinors have been studied in a previous paper[1]. This allows one to investigate the Dirac equation from a new point of view. Although the question of which of the relativistically invariant electron equations is to be considered correct remains unanswered[2], nevertheless such an investigation remains interesting in any case. For example, it will make possible the comparison of the Dirac equation with other types, thus allowing for a better understanding of their specific properties and differences. Since a great number of works have been devoted to the Dirac equation, comparison with them can help in the investigation and solution of other equations. On the other hand, the question of the meaning and nature of the Dirac equation may also be posed, which necessitates a thorough investigation of particular situations connected with it*.

1. First, let us note the well-known fact that if two systems of four-by-four matrices are given which satisfy the relations

$$\frac{1}{2}(R^\alpha R^\beta + R^\beta R^\alpha) = g_0^{\alpha\beta}, \tag{1}$$

$$(g_0^{11} = g_0^{22} = g_0^{33} = -g_0^{44} = 1,\ g_0^{\alpha\beta} = 0 \text{ for } \alpha \neq \beta),$$

$$\frac{1}{2}(\tilde{R}^\alpha \tilde{R}^\beta + \tilde{R}^\beta \tilde{R}^\alpha) = g_0^{\alpha\beta}, \tag{2}$$

then these matrices are equivalent, i.e.,

$$\tilde{R}^\alpha = O^{-1} R^\alpha O. \tag{3}$$

(The proof can be found, for example, in references 3 to 5.) The R^α, here, can of course have complex elements.

We emphasize particularly the meaning of transformations of the form of Eq. (3). In previous works[6,7] we treated the quantities R^α as matrices, corresponding to normalized basis vectors in four-dimensional pseudo-Euclidean space. Instead of the R^α, however, we can just as well use the matrices $\tilde{R}^\alpha$, since in the theory of matrix tensors the existence only of relations such as (1) is important. Here we shall deal with a different law of correspondence: to the same basis vectors will correspond matrices of a different form, though the basis vectors themselves remain invariant. We note in particular that one must not confuse a transformation from one coordinate system to another with a transformation from one isomorphic correspondence between matrices and four-vectors to another. These are altogether different things, though in both cases the formulas for expressing the new R^α in terms of the old can have the same form [e.g., Eq. (3)].

2. In Dirac theory, $m_0 c/\hbar$ is considered a scalar. It follows from this that the equation

$$R^\alpha \frac{\partial}{\partial x^\alpha} \psi_{(w)} = \nabla \psi_{(w)} = \frac{m_0 c}{\hbar} J \psi_{(w)}, \tag{4}$$

* For example, it necessitates consideration from a new point of view of the generalization of the Dirac equation to the case of the general theory of relativity. The author intends to devote a future paper to the investigation of real spinors in curvilinear coordinates and pseudo-Riemanian spaces, which may lead to the solution of this problem.

1 G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 29, 166 (1955).

2 G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 28, 530 (1955); Soviet Phys. 1, 491 (1955)

3 H. Weyl, *Gruppentheorie und Quantenmechanik*, 2nd ed., Leipzig, 1931.

4 B. L. Van der Waerden, *Group Theoretical Methods in Quantum Mechanics*

5 W. Pauli, *General Principles of Wave Mechanics*

6 G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 25, 667 (1953)

7 G. A. Zaitsev, J. Exper. Theoret. Phys. USSR 28, 524 (1955); Soviet Phys. 1, 411 (1955)

(where the matrices R^{α} are real and $\psi_{(w)}$ is a real spinor) is relativistically invariant.

The Dirac equation for the electron in the presence of an electromagnetic field can be written in the form

$$\left(i\beta \frac{\partial}{\partial x^4} + i\beta\alpha^k \frac{\partial}{\partial x^k}\right)\psi = \frac{m_0 c}{\hbar}\psi \quad (x^4 = ct,\ \alpha^k = \alpha_k) \tag{5}$$

(see Pauli[5], Sokolov[8], etc.). We will show that this reduces to two equations of type (4). To do this, let

$$i\beta = \tilde{R} = \tilde{R}^1\tilde{R}^2\tilde{R}^3,\ -i\beta\alpha^k = \tilde{J}\tilde{R}^k; \tag{6}$$

$$(\tilde{J} = \tilde{R}^1\tilde{R}^2\tilde{R}^3\tilde{R}^4,\ k = 1,2,3),$$

so that Eq. (5) becomes

$$\tilde{R}^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} = \frac{m_0 c}{\hbar}\tilde{J}\psi. \tag{7}$$

Transforming to real matrices R^{α} according to Eq. (3), and putting

$$O\psi = \psi_{(1)} + i\psi_{(2)} \tag{8}$$

($\psi_{(1)}$ and $\psi_{(2)}$ are column vectors of real elements), we get

$$R^{\alpha}\frac{\partial}{\partial x^{\alpha}}(\psi_{(1)} + i\psi_{(2)}) = \frac{m_0 c}{\hbar}J(\psi_{(1)} + i\psi_{(2)}). \tag{9}$$

The matrices α_k and β, and therefore $\tilde{R}^k$ and $i\tilde{R}^4$, are usually chosen Hermitian (see, for example, Pauli[9]). If R^k and iR^4 are chosen according to Table 1 of reference 6, they will also be Hermitian. Making use of the fact that any matrix which commutes with all the R^{α} is a multiple of the unit matrix, and taking the Hermitian conjugate of both sides of Eq. (3), we get $OO^* = O^*O = kE$. Choosing an appropriate multiplier for O, we get

$$O^* = O^{-1} \tag{10}$$

By way of an example of transition to real quantities, we consider the case of $R^k = IB(0)S_k$, $R^4 = IB(0)$, $J = I$, $R = B(0)$, and use the usual forms,

$$\alpha_1 = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix},\quad \alpha_2 = \begin{pmatrix} 0&0&0&-i\\ 0&0&i&0\\ 0&-i&0&0\\ i&0&0&0 \end{pmatrix}, \tag{11}$$

$$\alpha_3 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix},\quad \beta = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}$$

It is easy then to prove that (12)

$$O = \frac{1}{\sqrt{8}}\begin{pmatrix} 1-i & -1-i & 1-i & 1+i\\ 1+i & 1-i & -1-i & 1-i\\ -1-i & 1-i & -1-i & -1+i\\ -1+i & -1-i & 1-i & -1-i \end{pmatrix},$$

$$O^* = O^{-1} = \frac{1}{\sqrt{8}}\begin{pmatrix} 1+i & 1-i & -1+i & -1-i\\ -1+i & 1+i & 1+i & -1+i\\ 1+i & -1+i & -1+i & 1+i\\ 1-i & 1+i & -1-i & -1+i \end{pmatrix},$$

so that $\alpha_k = O^{-1}S_k O$, $i\beta = \tilde{R} = O^{-1}B(0)O$.

We shall show that $\psi_{(1)}$ and $\psi_{(2)}$ can be considered real spinors such as those considered in reference 6. It is first necessary to show that if $\psi_{(1)}$ and $\psi_{(2)}$ transform like real spinors, and ψ like a Dirac-theory spinor, then both sides of Eq. (8) transform in the same way. We shall make use, for the proof, of the notation used by Sokolov[8], so that

$$\beta = \alpha_0 = \rho_3,\quad \tilde{R}^k = \alpha_0\sigma_k, \tag{13}$$

$$\tilde{R}^4 = i\rho_2,\quad \tilde{J} = i\rho_1.$$

We shall also borrow from that reference certain formulas involving ψ. We first consider a spatial rotation. Rotation about the z-axis is characterized in Dirac theory by the matrix $\cos\phi/2 - i\sigma_3 \sin\phi/2 = \cos\phi/2 - \tilde{R}\tilde{R}_3 \sin\phi/2$ [here ϕ is the angle of rotation about the z-axis in the positive direction; we have used Eq. (18.11) on p. 96 of Sokolov[8], taking into account Eq. (13)]. Real spinors transform in the same way (see the remarks in reference 6 before Eq. (9); only $\tilde{R}^{\alpha}$ must be changed to R^{α}). Analogous considerations hold true for Lorentz transformations [this follows from a comparison of Eqs. (9)-(13) of reference 6 with those on pp. 93-95 of reference 8]. From this it follows that under a four-dimensional rotation, both sides of Eq. (8) transform in the same way. This will be true also for four-dimensional reflections, if the law of reflection of space or time is defined according to one of the last two cases of Pauli[9]. In this way it becomes possible to consider $\psi_{(1)}$ and $\psi_{(2)}$ as real spinors.

Equation (9) can be broken up into two parts by equating the real and imaginary parts. Since $\psi_{(1)}$

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[8] A. A. Sokolov and D. D. Ivanenko, *Quantum Theory of Matter*, GITTL, 1952

[9] W. Pauli, *Relativistic Theory of Elementary Particles*

and $\psi_{(2)}$ do not transform into each other under any four-dimensional rotations or reflections, Eq. (9) determines two equations for two unrelated real spinors $\psi_{(1)}$ and $\psi_{(2)}$, which can be written in the form

$$R^{\alpha}\,\frac{\partial\psi_{(w)}}{\partial x^{\alpha}} = \frac{m_0 c}{\hbar} J\psi_{(w)},\ w = 1,2. \qquad (14)$$

In this way it is seen that the Dirac equation for empty space is a combination of two independent equations containing separate real spinors, joined into one equation in a purely formal way.

3. Let us go on to a consideration of the general Dirac equation in the presence of an arbitrary external electromagnetic field. As is well-known, this equation (or, more accurately, system of equations) can be written in the form

$$\tilde{R}^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}} + i\,\frac{e}{\hbar c}\,A_{\alpha}\right)\psi = \frac{m_0 c}{\hbar}\tilde{J}\psi. \qquad (15)$$

Here e is the electronic charge and $A = -\phi$.

Taking into account Eqs. (3) and (8), we can rewrite Eq. (15) in the form of a system of equations containing real spinors:

$$R^{\alpha}\frac{\partial\psi_{(1)}}{\partial x^{\alpha}} - \frac{e}{\hbar c}\,A_{\alpha}R^{\alpha}\psi_{(2)} = \frac{m_0 c}{\hbar}\,J\psi_{(1)}, \qquad (16)$$

$$R^{\alpha}\,\frac{\partial\psi_{(2)}}{\partial x^{\alpha}} + \frac{e}{\hbar c}\,A_{\alpha}R^{\alpha}\psi_{(1)} = \frac{m_0 c}{\hbar}\,J\psi_{(2)}. \qquad (17)$$

This system of equations is clearly exactly equivalent to the Dirac equation.

Equation (14) is a gauge invariant. Thus, Eqs. (16) and (17) do not change their form under a spinor transformation of the second kind ($\psi_{(+)}$ is replaced by $e^{-if}\psi_{(+)}$), so long as A_{α} is replaced by $A_{\alpha} + \dfrac{\hbar c\,\partial f}{e\,\partial x^{\alpha}}$. In Dirac theory, the only tensor components which are bilinear forms in $\psi_{(1)}$ and $\psi_{(2)}$ that have physical meaning are those which are invariant under a spinor transformation of the second kind. Therefore, according to the preceding article[1], we can explain this situation by noting that the initial primary quantities are Ω_1, Ω_2 and the components $P_{(+)}$, N, and $F_{(+)}$*. Making use of the general theory of tensors, we can derive relations between the quantities under consideration (see reference 1), which, in particular, allow us to express $F_{(+)}$ in terms of $P_{(+)}$, N, Ω_1 and Ω_2. Some of these relations were previously known (see, for example, DeBroglie[10]), but their true meaning remained unclear.

Equations developed in the preceding article[1], which allow one to find the real spinors $\psi_{(1)}$ and $\psi_{(2)}$ in terms of the primary quantities, can be written in the form

$$P_{(+)}\psi_{(+)} = P_{(+)}(\psi_{(1)} + i\psi_{(2)}) = 2iN\psi_{(+)} \qquad (18)$$

and, consequently,

$$F_{(+)}\psi_{(+)} = -2i\,(\Omega_1 - \Omega_2 J)\,\psi_{(+)}. \qquad (19)$$

We can, of course, go back from $\psi_{(+)}$ to ψ, which corresponds to a transformation to a new isomorphic correspondence between matrices and tensors. Then, for example, instead of Eq. (19), we have $O^{-1}F_{(+)}O\,\psi = -2i\,(\Omega_1 - \Omega_2\tilde{J})\,\psi$, where $O^{-1}F_{(+)}O$ is very simply expressed in terms of the matrices α_k, β and the components of the tensor $F_{(+)}$.

In accordance with the above, the spinors which occur in the Dirac equation must be considered as secondary quantities. Therefore, we arrive at the conclusion that the Dirac equation must be considered a system of equations for definite four-dimensional tensors.

We must note that the notation we are using for writing the Dirac equation only with matrices and real elements is not new. The same notation was used, for example, by Majorana[11] (see also Pauli[9], Kramers[12], Markov[13], etc.). What is new is that the real spinors in this equation are considered secondary quantities, a system of parameters characterizing definite four-dimensional tensors. Accordingly, in the Dirac equation written in any form, the components of ψ must be considered secondary quantities defined by certain four-dimensional tensors.

4. In conclusion, we make some remarks concerning the difference between the Dirac equation and relativistically invariant differential equations of the first degree, containing two real spinors, which were discussed in reference 2.

The basic difference lies in the fact that these

* Also, the components of the external electromagnetic field and tensors which are invariant under spinor transformation of the second type and expressed in terms of the A_{α}, are components of real spinors and their derivatives.

[10] Louis de Broglie, *The Magnetic Electron*, 1936

[11] E. Majorana, Nuvo Cimento **14**, 171 (1937)

[12] H. A. Kramers, Proc. Amst. Acad. Sci. **40**, 814 (1937)

[13] M. A. Markov, J. Exper. Theoret. Phys. USSR **21**, 761 (1951)

equations are invariant under different kinds of spinor transformations. In connection with this, in each case different tensors are considered primary. This difference is closely connected with the fact that, in the earlier work[2], $\hbar$ is considered a pseudoscalar, not merely a scalar, as in Dirac theory, and that the operators of four-"momentum" have entirely different forms.

The different character of the two systems of differential equations is especially explicit in the transition to the nonrelativistic limit. From the point of view of the earlier work, we deal only with one real spinor, $\psi_{(1)}$. It is very characteristic that the current vector used in nonrelativistic quantum mechanics turns out to be not part of a vector, but of a tensor. Its components are proportional to $T^{(1)}_{4k}$ [see reference 1, Eq. (54) and Zaitsev[14], Eq. (59)].

As for the Dirac equation, in the transition to the equations of nonrelativistic mechanics, the situation is entirely different. In the nonrelativistic limit, ψ is still expressed in terms of two real spinors $\psi_{(1)}$ and $\psi_{(2)}$. The components of the current vector are found from the components of $P_{(+)}$ after making use of the Dirac equation and eliminating some of the terms (see Pauli[5])

[14] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 653 (1953)

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The Statistics of Charge-Conserving Systems and Its Application to the Theory of Multiple Production

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The quantum statistics of systems with a variable number of non-interacting particles is generalized to the case of an aggregate of oppositely charged particles, which obey the law of charge conservation. Formulas which differ from the corresponding formulas of ordinary quantum statistics are derived for the total number of particles and the total energy. The results obtained are applied to the theory of multiple production of mesons. The following questions are studied: the dependence of the energy on the relative proportions of neutral and charged mesons, the formation of nucleon-antinucleon pairs, and the relation between the yield and the primary energy. The theory is compared with the available experimental data.

1. INTRODUCTION

IN the statistical treatment of the phenomenon of multiple production of particles at high energies, proposed by Fermi[1], the total number of particles, the total energy of the system, and also the relation between the numbers of particles of different sorts in the "thermodynamic" approximation are calculated by the usual quantum statistical formulas for an ideal Bose or Fermi gas with a variable number of particles. However, in this case, it is more appropriate to use formulas which take into account the conservation of charge(electronic, nuclear, etc). This is particularly important when we consider processes with a low yield. Thus, after generalizing ordinary quantum statistics to the case of charge-conserving systems, a more detailed examination of processes of multiple production in the framework of the "thermodynamic" approximation is possible.

We make this generalization in the present paper, and as a result obtain new formulas for the total number of particles and the total energy, which we relate to the corresponding formulas of ordinary statistics. The results obtained are used to explain several matters pertaining to the theory of multiple production of particles.

[1] E. Fermi, *Elementary Particles*, New Haven, 1951

2. CALCULATION OF THE PARTITION FUNCTION, THE AVERAGE NUMBER OF PARTICLES, AND THE AVERAGE ENERGY OF CHARGE CONSERVING SYSTEMS

We shall consider an ideal gas, consisting of