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The Effect of Interelectronic Collisions on the Electrical Conductivity and Skin Effect in Metals

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The effect of collisions between electrons on the electrical conductivity, thermal conductivity and skin effect in metals is considered. The question of electronic viscosity is discussed. It is shown that taking account of interelectronic collisions does not introduce notable changes into the results of the theory of the anomalous skin effect in metals.

IN a recent article[1] the assertion was made that the electronic viscosity in metals, considered as the result of interelectronic interaction, is essential in the calculation of the absorbing power of metals in the infrared region of the spectrum. If this assertion were true, then it would be of undoubted interest, especially in view of the problems currently confronting the optics of metals[2]. Hence we have attempted to consider in more detail the question of the effect of the interelectronic interaction on the optical properties of metals. This is especially necessary in connection with the series of unclear or doubtful moments present in reference 1. The question considered here is also closely connected with the question of the influence of the interelectronic interaction on the electrical conductivity and other kinetic coefficients. Thus in Part 1 of the present work, the effect of interelectronic collisions on electric and thermal conductivity is considered, in Part 2, the problem of electronic viscosity is considered and, finally Part 3 is devoted to the explanation of the effect of collisions between electrons on the skin effect in metals, and, in particular, on their absorbing power. In addition to this it is shown that the method and the results of reference 1 are incorrect and that for frequencies which are not too high it is not essential that the interelectronic interaction be taken into account.

1. In the theory of the electrical conductivity of metals the interaction between electrons is not ordinarily taken into account explicitly, in spite of the fact that the corresponding energy of interaction is of the same order as the energy of interaction of the electrons with the lattice. Until now no strict theoretical basis for the possibility of successfully using such an approximation (in the case of metals not in a superconducting state) has been given. However, there is no doubt that the effect of the interaction between electrons is greatly weakened in view of the Pauli principle. The success of the shell representation in the case of nuclear theory is analogously explained. Of course, the operation of the Pauli principle cannot reduce the effect of the interelectronic interaction to exactly zero, since in the zone where the Fermi distribution is washed out electrons may change their state. Together with this it is natural to think that in this region the interelectronic interaction might be taken account of by the method of the kinetic equation and considered as a perturbation. In what follows we shall do just this, analogously to the way in which it has been done in earlier works[3-5], to explain the effect of interelectronic collisions on the statistical electrical conductivity. In order to facilitate the exposition and also in order to be able to make a few observations, we shall first dwell further on this question.

Within the framework of elementary representations the length of the free path for interelectronic collisions can be written directly

$$l_{ee} \sim \frac{1}{q_{ee} n_0 (kT/E_0)^2}, \qquad \nu_{ee} = \frac{1}{\tau_{ee}} = \frac{v_0}{l_{ee}} \sim q_{ee} n_0 \left(\frac{kT}{E_0}\right)^2 v_0, \tag{1}$$

where n_0 is the concentration of conduction electrons, $E_0 = \frac{1}{2} m v_0^2$ is the energy at the Fermi boundary and q_{ee} is the corresponding effective cross section. (For electrons which are not free the role of n_0/E_0 is played by the density of states $2\,dN/dE$). The meaning of the expressions (1) becomes completely clear if we take account

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[1] C. W. Benthem and R. Kronig, Physica **20**, 293(1954)

[2] V. L. Ginzburg, Dokl. Akad. Nauk SSSR **97**, 999 (1954)

[3] L. D. Landau and I. Ia. Pomeranchuk, J. Exper. Theoret. Phys. USSR **7**, 379 (1937)

[4] W. G. Baber, Proc. Roy. Soc.(London) **A158**, 383 (1937)

[5] S. V. Vonsovskii and A. A. Berdyshev, J. Exper. Theoret. Phys. USSR **25**, 723 (1953)

of the fact that an electron lying in the zone where the Fermi distribution washes out can collide only with one of the electrons in that same region, but the concentration of such electrons is $\sim n_0(kT/E_0)$. A second factor of kT/E_0 appears in Eq. (1) in virtue of the Pauli principle being taken into account for the final state (both electrons must also be in the washed out zone after the collision).

For purposes of estimation we may take as q_{ee} either the Coulomb cross section $q'_{ee} = (\pi e^4/E_r^2) \ln (E_r\rho/e^2)$ or the cross section $q''_{ee} \sim \pi n_0^{-2/3}$, where E_r is the energy of the relative motion and ρ is the radius of cutoff (of screening). Since $\rho \sim n_0^{-1/3} \sim 3 \times 10^{-8}$ cm and $E_r \sim E_0 \sim 10^{-11} - 10^{-12}$erg, then $q'_{ee} \sim \pi e^4/E_0^2 \lesssim 3 \times 10^{-15}\text{cm}^2$ and $q''_{ee} \sim 3 \times 10^{-15}\text{cm}^2$. The numerical calculations carried out by Abrahams[6] for $E_0 \approx 3.5$ ev give $q_{ee} \approx 1.5 \times 10^{-15}\text{cm}^2$. Taking, for purposes of orientation, $q_{ee} \approx 10^{-15}\text{cm}^2$, $n_0 \sim 3 \times 10^{22}\text{cm}^{-3}$ and $E_0 \sim 3$ev (that is, $T_0 = E_0/k \sim 3 \times 10^4$ degrees), we obtain

$$l_{ee} \sim 3 \cdot 10^{-8} (E_0/kT)^2 \sim 3 \cdot 10^{-3} (10^2/T)^2 \text{ cm} \quad (2)$$

For collisions of electrons with the lattice we find, if we take the path length $l_{el} \sim 3 \times 10^{-6}$ cm for $T \sim 300°$,

$$l_{el} \sim \frac{10^{-3}}{T} \text{ cm} \quad (\text{for} \quad T \gg \Theta \sim 10^2),$$

$$l_{el} \sim 10^{-6} \left(\frac{10^2}{T}\right)^5 \text{ cm} \quad (\text{for} \quad T \ll \Theta \sim 10^2). \quad (3)$$

Values of l_{ee} and l_{el} are given in Table 1, along with the path length for collisions with impurities, which is taken, by way of example, as $l_{ei} \sim 3 \times 10^{-3}$cm (this corresponds to a very pure specimen).

TABLE I

T, degree	l_{ee}, cm	l_{el}, cm	l_{en}, cm
10^3	$3 \cdot 10^{-5}$	10^{-6}	$3 \cdot 10^{-3}$
300	$3 \cdot 10^{-4}$	$3 \cdot 10^{-6}$	$3 \cdot 10^{-3}$
100	$3 \cdot 10^{-3}$	—	$3 \cdot 10^{-3}$
30	$3 \cdot 10^{-2}$	$3 \cdot 10^{-4}$	$3 \cdot 10^{-3}$
10	0.3	0.1	$3 \cdot 10^{-3}$
1	30	10^4	$3 \cdot 10^{-3}$

From Table 1 it is clear that, beginning at 10 – 20° and below, interelectronic collisions may play a notable, or even a determining, role in the temperature dependent part of the resistance, which part is proportional to $(1/l_{el}) + (1/l_{ee})$. Experimental results[7] show that for Na at $T = 8 - 10°$ the contribution of interelectronic collisions to the temperature dependent part of the resistance does not exceed 10 – 20% (we here make use of Mattissen's law, which is probably valid in the given case). This result does not contradict the estimates cited, but indicates that with further precision in the experiment and in the corresponding calculations there may be hope of finding an explanation of the role of interelectronic collisions, for example, in Na. Such an explanation would have great significance. The fact is that the estimates cited may reasonably be applied to the calculation of electrical conductivity only in case the interelectronic collisions lead to a change in the current. By virtue of the law of conservation of momentum this does not occur for free electrons, and thus an expression of the type of Eq. (1) can be used in estimating the change in electrical conductivity only on the presupposition of the presence of exchange processes[3] or semiconductivity[4], when an electron collides with a "hole". Within the framework of existent representations the latter process cannot occur in univalent metals. Exchange processes for collisions of electrons with phonons also cannot occur* in this case, while exchange processes for interelectronic collisions can occur only very seldom, for definite favorable conditions (see below). Hence it seems appropriate to estimate the effect of interelectronic collisions on the electrical conductivity both with account being taken of exchange processes and without account being taken of them. This is done below by the method of the kinetic equation.

Writing the distribution function in the form $f = f_0 + \phi$, where f_0 is the equilibrium (Fermi) function, we obtain in the linear approximation

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[6] E. Abrahams, Phys. Rev. **95**, 839 (1954)

[7] D. K. C. MacDonald and K. Mendelssohn, Proc. Roy. Soc. (London) **A202**, 103 (1950)

*The question as to whether these processes actually occur is not completely clear, but according to Klemens[8] (see also Wilson[9]) there are bases for supposing that a stable state of the lattice may perhaps be established even without exchange processes.

[8] P. G. Klemens, Proc. Phys. Soc. (London) **A64**, 1030 (1951)

[9] A. H. Wilson, *The Theory of Metals*, (Cambridge, 1953)

$$i\omega\varphi + \frac{\mathbf{p}}{m}\frac{\partial\varphi}{\partial\mathbf{r}} - e\vec{\mathcal{E}}\frac{\partial f_0}{\partial\mathbf{p}} = -J_{el} - J_{ee}, \qquad (4)$$

where the field $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}e^{i\omega t}$, m is the effective mass of the electron, J_{el} is the integral of the collisions of the electrons with the lattice (with phonons), which is presumed to be in equilibrium at a certain temperature T, and J_{ee} is the integral of the collisions of the electrons with each other.

Concerning ourselves with the electrical conductivity, we seek a solution of Eq. (4) in the form:

$$\varphi = e\vec{\mathcal{E}}\mathbf{p}A(p). \qquad (5)$$

Relation (5) leads to the following expression for the total current density

$$j_t = j + i\omega P$$

$$= \sigma'(\omega)\mathcal{E} \equiv \left(\sigma + \frac{\varepsilon - 1}{4\pi}i\omega\right)\mathcal{E} = \frac{i\omega}{4\pi}(\varepsilon' - 1)\mathcal{E},$$

where j is the conduction current density, P the polarization, σ' the complex electrical conductivity, ϵ the dielectric constant, σ the electrical conductivity and $\epsilon' = \epsilon - i(4\pi\sigma/\omega)$ the complex dielectric constant. The magnitudes of ϵ, σ and ϵ' depend on ω; we may also designate by σ_0 the magnitude of the statistical electrical conductivity $\sigma'(0) = \sigma(0)$.

Considering a field not very different from homogeneous, we neglect the space derivatives of the electric field which arise when Eq. (5) is substituted into Eq. (4). The following equation, determining $A(p)$, is then obtained from Eq. (4):

$$\partial f_0/\partial\mathbf{p} = J_{ee}[\mathbf{p}A(p)] + [\nu_{el}(p) + i\omega]\,\mathbf{p}A(p). \qquad (6)$$

As a result of Eq. (6) it is considered that in the high and low temperature regions which are of interest to us:

$$J_{el}[\mathbf{p}A(p)] = \nu_{el}(p)\,\mathbf{p}A(p), \qquad (7)$$

where the number of collisions with the lattice (with phonons and impurities) $\nu_{el}(p)$ is a certain function of p.

Equation (6) is somewhat complicated. However, the situation becomes essentially simplified in the case where the interaction of the electrons with each other may be considered weak, that is, where it changes the magnitudes of σ and ϵ only slightly. Then, applying the method of successive approximations, one can obtain the following expression for $A(p)$:

$$A(p) = \frac{\mathbf{p}\,\partial f_0/\partial\mathbf{p}}{p^2[\nu_{el}(p) + i\omega]}$$

$$- \frac{p_i}{p^2[\nu_{el}(p) + i\omega]}J_{ee}\left(\frac{p_i\,p_j\,\partial f_0/\partial p_j}{p^2[\nu_{el}(p) + i\omega]}\right). \qquad (8)$$

Neglecting collisions of electrons with each other, that is, neglecting J_{ee}, and assuming that f_0 depends only on the energy E of the electron, we find that Eq. (8) leads to the well-known formula for σ'

$$\sigma' = \sigma'_{el} = \frac{2e^2}{3}\frac{v_0^2}{\nu_{el}(p_0) + i\omega}\left(\frac{dN}{dE}\right)_{E_0}$$

$$= \frac{e^2 n_0}{m[\nu_{el}(p_0) + i\omega]}, \qquad (9)$$

where $2\,dN/dE$ is the density of states and n_0 is the concentration of conduction electrons.

The change in σ' arising from the interaction of the electrons with each other is determined by an integral of the form:

$$I = \frac{2}{(2\pi\hbar)^3}\int d\mathbf{p}_1\,\Phi(\mathbf{p}_1)\,J_{ee}\left[\psi(\mathbf{p}_1)\frac{\partial f_0}{\partial E_1}\right], \qquad (10)$$

where ψ and Φ are certain functions of the electron's momentum $\mathbf{p}_1$[the form of the function ψ is clear from Eq. (8)], E_1 is the energy of the electron and the integral of the collisions of electrons with electrons has the form:

$$J_{ee}\left[\psi(\mathbf{p}_1)\frac{\partial f_0}{\partial E_1}\right] \qquad (11)$$

$$= \frac{1}{kT}\sum_n\int d\mathbf{p}_2\,d\Omega_{\mathbf{p}_1'}\,q_{ee}(|\mathbf{p}_1 - \mathbf{p}_2|, \vartheta)\left|\frac{\mathbf{p}_1 - \mathbf{p}_2}{m}\right|$$

$$\times f_0(p_1'^2)\,f_0(p_2'^2)\,[1 - f_0(p_1^2)]\,[1 - f_0(p_2^2)]$$

$$\times\ [\psi(\mathbf{p}_1) + \psi(\mathbf{p}_2) - \psi(\mathbf{p}_1') - \psi(\mathbf{p}_2')],$$

where $q_{ee}(p, \vartheta)$ is the cross section for collisions of electrons with electrons, for which ϑ is the angle characterizing the scattering. From the laws of conservation of energy and quasi-momentum of the electrons**

$$\mathbf{p}_2' = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - 2\pi\mathbf{g}\,n. \qquad (12)$$

$$p_1'^2 = p_1^2 + p_2^2 - (\mathbf{p}_1 + \mathbf{p}_2 - 2\pi\mathbf{g}\,n)^2,$$

** It is assumed that during the scattering the electrons remain in the same zone. The energy is regarded as a quadratic function of the momentum ($E = p^2/2m_{\text{eff}}$).

where $\mathbf{g}/\hbar$ is the vector of the reciprocal lattice and n is an integer.

We shall consider integral (10) without specifying the form of the functions Φ and ψ for the present, since we have also in view using the resulting formulas in a different way than in their application to the calculation of the electrical conductivity. Taking into account Eqs. (11) and (12), we write Eq. (10) in the following form

$$\begin{aligned}
I = {} & \frac{4}{(2\pi\hbar)^6\, mkT}\int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{s} \\
& \times q_{ee}(|\mathbf{p}_1-\mathbf{p}_2|, \cos\vartheta)\,|\mathbf{p}_1-\mathbf{p}_2|\,[1-f_0(p_1^2)] \\
& \times [1-f_0(p_2^2)]\, f_0\Big\{\frac{p_1^2+p_2^2}{2} \qquad (13) \\
& \qquad + \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2} \\
& \times |\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n|\cos\vartheta\Big\} \times f_0\Big\{\frac{p_1^2+p_2^2}{2} \\
& \qquad - \frac{1}{2}\sqrt{2(p_1^2+\mathbf{p}_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2} \\
& \times |\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n|\cos\vartheta\Big\}\,\Phi(\mathbf{p}_1) \\
& \qquad \times \Big\{\psi(\mathbf{p}_1)+\psi(\mathbf{p}_2)-\psi\Big(\frac{\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n}{2} \\
& + \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2}\,\mathbf{s}\Big) \\
& - \psi\Big(\frac{\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n}{2} \\
& \qquad - \frac{1}{2}\sqrt{2(p_1^2+p_2^2)-(\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n)^2}\,\mathbf{s}\Big)\Big\}.
\end{aligned}$$

Here $\mathbf{s}$ is a unit vector, ϑ is the angle between $\mathbf{s}$ and $\mathbf{p}_1+\mathbf{p}_2-2\pi\mathbf{g}n$. In the integration with respect to $\cos\vartheta$ the principal contribution to the integral is made by the region $\cos\vartheta \sim 0$; hence in the slowly varying functions we may set $\cos\vartheta = 0$ and extend the integral with respect to $y = \cos\vartheta$ from $-\infty$ to $+\infty$; in the same way we can set $p_1 = p_2 = p_0$ in the slowly varying functions. We thus obtain

$$\begin{aligned}
I = {} & \frac{32\,\zeta(3)}{(2\pi)}\,\frac{m^2 p_0^2 (kT)^2}{\hbar^6}\int d\Omega_{\mathbf{p}_{10}}\, d\Omega_{\mathbf{p}_{20}}\int_0^{2\pi} d\varphi_{\mathbf{s}_0} \times \\
& \times \frac{q_{ee}(|\mathbf{p}_{10}-\mathbf{p}_{20}|, 0)\,|\mathbf{p}_{10}-\mathbf{p}_{20}|}{|\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n|\sqrt{4p_0^2-(\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n)^2}} \\
& \times \Phi(\mathbf{p}_{10})\Big[\psi(\mathbf{p}_{10})+\psi(\mathbf{p}_{20})-\psi\Big(\frac{\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n}{2} \\
& + \frac{1}{2}\sqrt{4p_0^2-(\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n)^2}\,\mathbf{s}_0\Big) \qquad (14) \\
& - \psi\Big(\frac{\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n}{2} \\
& \qquad - \frac{1}{2}\sqrt{4p_0^2-(\mathbf{p}_{10}+\mathbf{p}_{20}-2\pi\mathbf{g}n)^2}\,\mathbf{s}_0\Big)\Big] \\
& + O\Big[\Big(\frac{kT}{E_0}\Big)^3\Big],
\end{aligned}$$

where $\mathbf{s}_0 = \mathbf{s}(\cos\vartheta = 0)$ and $\zeta(3) = 1.202$ [$\zeta(z)$ is the Riemann function].

If exchange processes are neglected or if they are impossible, when $n = 0$ in Eq. (12), the principal member of I, proportional to T^2, is equal to zero. As a result, the change in electrical conductivity appears to be reduced by a factor $\sim kT/E_0$. In the interesting temperature region $T \sim 10°$ this factor is of the order of 3×10^{-4} (for $T_0 = E/k \sim 3 \times 10^{-4}$), and the effect of interelectronic collisions on σ' can be considered practically equal to zero[‡]. Exchange processes for collisions of the electrons are possible according to Eq. (12) only if $4p_0 \geq 2\pi g n$, since all the momenta p_1, p_2, p_1' and p_2' are near p_0. In the one electron model for a univalent metal, half the conduction zone is filled and $4p_0 = 2\pi g$, that is, for $n = 1$ exchange processes are still possible but are already very difficult. Assuming that exchange processes occur only for $n = 1$, we obtain,

$$\frac{\sigma' - \sigma'_{el}}{\sigma_{el}} \equiv \frac{\Delta\sigma'_{ee}}{\sigma'_{el}} = -\frac{\nu_{ee}}{\nu_{el} + i\omega}, \qquad (15)$$

$$\nu_{ee} = q_{\text{eff}}\, n_0 v_0 \Big(\frac{kT}{E_0}\Big)^2 = \frac{v_0}{l_{ee}},$$

where $\Delta\sigma'_{ee}$ is the change in σ' connected with interelectronic collisions, $\nu_{el} = v_0/l_l$ is the number of collisions with the lattice $\left(\frac{1}{l_l} = \frac{1}{l_{el}} + \frac{1}{l_{ei}}\right)$ and

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[‡] We note[10] that for a non-degenerate electron-ion plasma, taking account of the interelectronic collisions changes the statistical electrical conductivity by an amount of the order of unity (σ_0 increases by a factor of 1.73). At high frequencies, when $\omega^2 \gg \nu^2$, the effect of interelectronic collisions is small even in the case of a plasma[11].

[10] R. Landshoff, Phys. Rev. **76**, 904 (1949); **82**, 442 (1951)

[11] V. L. Ginzburg, J. Tech. Phys. USSR **21**, 943 (1951)

$$q_{\mathrm{eff}} = \frac{3\pi^2 \zeta(3)}{x} \int_{x-1}^{1} dz q_{ee}(2p_0\sqrt{1-(x-z)^2}, 0)$$

$$\times \sqrt{1-(x-z)^2}$$

$$\times \left\{\frac{\pi}{2} - \arcsin z + z\sqrt{1-z^2}\right\},$$

where $2 \geq x = \pi g/p_0 \geq 1$.

To the same accuracy we may also write (assuming that $\nu_{ee}/ |\nu_{el} + i\omega| \ll 1$)

$$\sigma' = e^2 n_0 / (m[\nu_{el} + \nu_{ee} + i\omega]). \tag{16}$$

If exactly half the zone is filled, then $x = 2$, $q_{\mathrm{eff}} = 0$ and exchange processes may be said to occur only in the next approximation in terms of kT/E_0. This result is connected also with the assumption of the sphericity of the Fermi surface and the quadraticity of the dependence of the energy on the momentum. The absence of a contribution to the resistance of Na from electron-electron collisions would seem to show that the usual isotropic one electron model applies well to this case.

In order to estimate the maximum possible contribution from the exchange processes, we set $x \approx 1$. Then for the hard sphere model $q_{ee}(\theta) = \rho^2/4$ (ρ is the radius of a sphere) and $q_{\mathrm{eff}} = 0.97 \times 3\pi^2\zeta(3)\rho^2/4 = 2.7\pi\rho^2$; for the screened Coulomb field $[V_{12} = (e^2/r_{12})e^{-r_{12}/\rho}]$

$$q_{\mathrm{eff}} = 3\pi^2\zeta(3)\frac{e^4}{16E_0^2} y\left(\frac{\hbar}{p_0\rho}\right),$$

where the magnitudes of y in their dependence on the parameter $\hbar/p_0\rho$ are arranged in Table 2.

TABLE II

$\hbar/p_0\rho$	y
0.2	1.64
0.5	0.78
1.0	0.36
2.5	0.10
5.0	0.03

Setting $\rho = 2 \times 10^{-8}$cm and $E_0 = 3$ev ($p_0 = 10^{-19}$), we obtain $q_{\mathrm{eff}} \approx 3 \times 10^{-15}$ in the case of the spheres, and $q_{\mathrm{eff}} \approx 4 \times 10^{-15}$ in the Coulomb case. In the estimates given earlier we set $q_{\mathrm{eff}} \equiv q_{ee} \sim 10^{-15}$ (see reference 2), and consequently the magnitudes of l_{ee} given in Table 1 are acceptable and may even be increased several fold. In the latter case, the effect of interelectronic collisions would already be notable in Na. Thus there already exist known bases for supposing that exchange processes do not play a major role in the case of Na. Later experiments may in principle lead to a solution of the question concerning the role of exchange processes in Na and other metals as a result of the determination of the temperature dependence of σ_0 for low temperatures. However, in virtue of the impossibility of relying on Mattissen's law, it is evident that success is possible here only for samples with negligibly small impurities, when $l_{ei} \lesssim 0.1$ cm (such samples have not yet been obtained).

The effect of interelectronic collisions on the thermal conductivity of metals is incomparably less than in the case of the electrical conductivity. This is clear from the fact that the path length for collisions of electrons with phonons, which enters into the expression for the coefficient of thermal conductivity, is less than the corresponding path length in the case of electrical conductivity by a factor $\sim (\Theta/T)^2$, where Θ is the Debye temperature. At the same time, the contribution of interelectronic collisions to electrical and thermal conductivity is determined by path lengths which are of the same order of magnitude. As a result, interelectronic collisions may be considered to occur only for $T \sim 0.1°$, but they are non-essential even in this region, apart from effects connected with the heat conductivity of the lattice, the finite dimensions of the samples or of the crystalline particles composing them, and the like.

In spite of the small role which, as is clear from the above, is played by interelectronic collisions, nevertheless, from the point of view of their effect on exchange processes, these collisions may be extremely important. Thus, in a consideration of nonlinear processes in metals, interelectronic collisions cannot be neglected, generally speaking, and, for example, at high temperatures they lead to the possibility of using the concept of the kinetic temperature of the electrons in a metal even for large currents[12].

2. During a change in the density of the electron current from point to point, a viscosity effect, similar to that which takes place during the flow of ordinary gases, may appear. The question of electronic viscosity is discussed in references 13,

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[12] V. L. Ginzburg and V. P. Shabanskii, Dokl. Akad. Nauk SSSR 100, 445 (1955)

[13] A. S. Kompaneets, J. Exper. Theoret. Phys. USSR 9, 920 (1939)

14 and 1, wherein the point of departure is essentially the equation of motion of the electrons:

$$mn_0\frac{d\mathbf{u}}{dt} + mn_0\nu\mathbf{u} - \eta_{ee}\Delta\mathbf{u} = en_0\vec{\mathcal{E}}, \tag{17}$$

where $\mathbf{u}$ is the average velocity of the electrons, η_{ee} is the coefficient of viscosity and $\nu \approx \nu_{el}$ is the effective number of collisions.

In the linear approximation and with the viscosity neglected, $\mathbf{j}_t = en_0\mathbf{u} = \frac{e^2 n_0\vec{\mathcal{E}}}{m(\nu_{el}+i\omega)} = \sigma'\vec{\mathcal{E}}$. Hence, considering the role of the viscosity to be small, we obtain from Eq. (17)

$$\mathbf{j}_t = \sigma'\vec{\mathcal{E}} + \xi_{ee}\Delta\vec{\mathcal{E}}, \qquad \xi_{ee} = \frac{\eta_{ee}\sigma'}{mn_0(i\omega+\nu_{el})} = \frac{e^2\eta_{ee}}{m^2(i\omega+\nu_{el})^2}. \tag{18}$$

We shall evaluate the coefficient of viscosity η_{ee}. Let the velocity $\mathbf{u}$ be directed along the x axis and be changing along the z axis. Then for $\omega = 0$, as follows from elementary kinetic considerations, the density of the current of momentum transferred to the lattice is of the order of $\frac{mn_0v_0}{3}l_l\frac{du}{dz} = \eta_{el}\frac{du}{dz}$, where $\frac{1}{l_l} = \frac{1}{l_{el}} + \frac{1}{l_{ei}}$ and the fact that $l_{ee} \gg l_l$ has been taken into account. We are now interested in the current of momentum transferred to the electrons, equal to $\eta_{ee}du/dz$. In order to find η_{ee}, it is evidently necessary to multiply the coefficient η_{el} (see above) by the number of collisions of the electrons with the lattice along the path l_l, that is, by l_l/l_{ee}. Thus (see also reference 1)

$$\eta_{ee} \sim \frac{mn_0v_0}{3}l_l\frac{l_l}{l_{ee}} \sim \frac{mv_0}{3q_{ee}}\left(\frac{E_0}{kT}\right)^2\left(\frac{l_l}{l_{ee}}\right)^2 = \frac{mv_0}{3q_{ee}}\beta \equiv \eta_0\beta. \tag{19}$$

Similarly, for arbitrary frequency

$$\eta_{ee} \sim \frac{mn_0v_0^2}{3}\frac{\nu_{ee}}{\omega^2+\nu_{el}^2}, \quad \nu_{ee} = \frac{v_0}{l_{ee}}, \quad \nu_{el} = \frac{v_0}{l_l}. \tag{20}$$

The magnitude of $\eta_0 = mv_0/3q_{ee} \sim 10^{-5}$ and may be written in the form[14] $\eta_0 \sim (3/8)n_0\hbar$, since $q_{ee} \sim n_0^{-2/3}$ and $n_0 \sim (mv_0/2\pi\hbar)^3$. Using the magnitudes of l_{ee} and l_l introduced in Part 1, we see that for high temperatures the coefficient $\beta \sim 1$ (for $T_0 = E_0/k \sim 3 \times 10^4$). The coefficient β increases as the temperature is lowered and reaches the magnitude $\beta \sim 10^3$ for $T \sim 20°$, but beyond this point it begins to fall and tends to zero for $T \to 0$. It follows from this that the use of references 14 and 1 of the constant magnitude $\eta_{ee} \sim \eta_0$ (that is, $\beta \sim 1$) is unsuitable in the region of low temperatures [+].

We shall not consider in further detail the question of the form of the function $N_{ee}(T)$, since the use of equations (17) – (18), where η_{ee} is the coefficient of viscosity connected with the interaction between electrons, is inadmissible. The fact is that even in the complete absence of interelectronic collisions, when $\eta_{ee} = 0$, in a slightly inhomogeneous field

$$\mathbf{j}_t = \sigma'\vec{\mathcal{E}} + \xi_{el}\Delta\vec{\mathcal{E}}, \tag{21}$$

where in the quasistatic case $\xi_{el} \sim \sigma l_l^2$. The appearance of the second term in Eq. (21) is explained by the fact that the current at a given point depends on the field in the region of this point within a radius of the order of the free path length, and the differential form of Ohm's law is correct only if $l_l^2/\delta^2 \to 0$, where δ is the skin depth (the distance in which the field $\mathcal{E}$ changes notably). In other words, the term of type $\xi\Delta\vec{\mathcal{E}}$ appears as soon as account is taken of the anomalous character of the skin effect[16,17],

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[14] R. Kronig and J. Korringa, Physica 10, 406, 800 (1943); R. Kronig, Physica 15, 667 (1949)

[+] In reference 14 the magnitude of η_{ee} is determined not from kinetic considerations but from an equation obtained by comparing the electron viscosity formula for ionization losses of energy at high frequencies in metals with an analogous formula found by another method[15]. Without pausing to go into this question in detail, we note that the use in reference 14 of an equation of the type of Eq. (17) is completely invalid (even apart from the necessity of taking account of collisions with the lattice, which is spoken below). This is already clear from the fact that in reference 14 the equation of type (17) is applied under conditions where the velocity is changing within distances of 10^{-8}- 10^{-10} cm, which are significantly smaller not only than l_{ee} but also than l_l, while, as is known from kinetic theory, Eq. (17) may be used only if the velocity $\mathbf{u}$ changes but slightly in distances of the order of the free path length.

[15] H. A. Kramers, Physica 13, 401 (1947)

[16] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) A195, 336 (1948)

[17] R. B. Dingle, Physica 19, 311 (1953)

without consideration of interelectronic collisions. If account be taken of these collisions, then it is only necessary to change the coefficient slightly, the corresponding correction being small in virtue of the fact that $l_{ee} \gg l_l$. Actually, when account is taken of the anomalous character of the skin effect and of interelectronic collisions

$$j_t = \sigma' \vec{\mathcal{E}} + \xi_{el} \Delta \vec{\mathcal{E}} + \xi_{ee} \Delta \vec{\mathcal{E}}, \qquad (22)$$

where in the quasistatic case $\xi_{el} \sim \sigma' l_l^2$ and $|\xi_{ee}| \sim \sigma' l_l^2 \frac{l_l}{l_{ee}} \ll \xi_l$ [see Eqs. (18) and (19)].

Everything that has been said is confirmed by a more rigorous kinetic consideration, on which it is also pertinent to dwell briefly in order to complete the picture.

Assuming that the electric field is slightly inhomogeneous, and assuming also that the spatial inhomogeneity in the distribution of the electrons is not large, we shall, following the method worked out in reference 18, seek a solution of Eq. (4) in the form:

$$\varphi = e \vec{\mathcal{E}} \mathbf{p} A(p) + e \frac{\partial \mathcal{E}_k}{\partial r_j} \left(\frac{p_k p_j}{p^2} - \frac{\delta_{jk}}{3} \right) B(p) \qquad (23)$$

$$+ e \frac{\partial^2 \mathcal{E}_k}{\partial r_i \partial r_j} \Big\{ C(p) p_k \delta_{ij} + D(p) (p_i \delta_{kj} + p_j \delta_{ki})$$

$$+ F(p) \Big[p_k \Big(p_i p_j - \frac{3}{5} p^2 \delta_{ij} \Big) + p_i \Big(p_j p_k - \frac{3}{5} p^2 \delta_{jk} \Big)$$

$$+ p_j \Big(p_k p_i - \frac{3}{5} p^2 \delta_{ki} \Big) \Big] \Big\} .$$

We shall neglect higher derivatives of the electric field. Relation (23) leads to the following expression for the total current*

$$\mathbf{j}_t = \sigma' \vec{\mathcal{E}} + \xi \Delta \vec{\mathcal{E}}, \qquad (22a)$$

where σ' is determined as a function of $A(p)$ (see reference 5), and ξ is determined by the formula

$$\xi = - \frac{e^2}{3m} \int p^2 d\mathbf{p}\, C(p). \qquad (24)$$

Putting Eq. (23) into Eq. (4) leads to Eq. (6) for $A(p)$ and to the following equations for $B(p)$ and $C(p)$:

$$\frac{p^2}{m} \left(\frac{p_j p_k}{p^2} - \frac{\delta_{jk}}{3} \right) A(p) = J_{ee} \left[\left(\frac{\delta_{jk}}{3} - \frac{p_j p_k}{p^2} \right) B(p) \right]$$
$$- \left(\frac{\delta_{jk}}{3} - \frac{p_j p_k}{p^2} \right) [\nu'_{el}(p) + i\omega]\, B(p), \qquad (25)$$

$$\frac{\mathbf{p}}{5m} B(p) \qquad (26)$$
$$= - J_{ee} [\mathbf{p}\, C(p)] - [\nu_{el}(p) + i\omega]\, \mathbf{p}\, C(p),$$

where $\nu_{el}(p)$ is determined by Eq. (17), while $\nu'_{el}(p)$ is determined in the following manner:

$$J_{el} \left[\left(\frac{\delta_{ij}}{3} - \frac{p_i p_j}{p^2} \right) B(p) \right] \qquad (27)$$
$$= \nu'_{el}(p) \left(\frac{\delta_{ij}}{3} - \frac{p_i p_j}{p^2} \right) B(p).$$

Assuming, as was done during the solution of Eq. (6), that the effect of the interaction of the electrons on each other may be considered as a perturbation, we find the following expression for $C(p)$:

$$C(p) = \frac{p^2 A(p)}{5m^2 [\nu_{el}(p) + i\omega][\nu'_{el}(p) + i\omega]} \qquad (28)$$
$$+ \frac{p_k}{5p^2 m^2 [\nu_{el}(p) + i\omega]} J_{ee} \left(\frac{p_k p^2 A(p)}{[\nu_{el}(p) + i\omega][\nu'_{el}(p) + i\omega]} \right)$$
$$- \frac{3 p_i p_k J_{ee} \left[\left(\frac{\delta_{ik}}{3} - \frac{p_i p_k}{p^2} \right) \mathrm{p} \frac{\partial f_0}{\partial \mathrm{p}} (\nu_{el} + i\omega)^{-1} (\nu'_{el} + i\omega)^{-1} \right]}{10 m^2 p^2 [\nu_{el}(p) + i\omega][\nu'_{el}(p) + i\omega]} .$$

It follows from this that if the interaction of the electrons on each other be neglected:

$$\xi_{el} = \frac{2e^2}{15} \frac{v_0^4 (dN/dE)_{E_0}}{[\nu_{el}(p_0) + i\omega]^2 [\nu'_{el}(p_0) + i\omega]} \qquad (29)$$
$$= \frac{e^2 n_0^2 p_0^2}{5m^3 [\nu_{el}(p_0) + i\omega]^2 [\nu'_{el}(p_0) + i\omega]} ;$$

$$\eta = \eta_{el} = \xi_{el} [i\omega + \nu_{el}(p)]^2 (m/e)^2 \qquad (30)$$
$$= \frac{n_0 p^2}{5m [\nu'_{el}(p) + i\omega]} .$$

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* Here and above it is assumed for simplicity that $\mathrm{div}\vec{\mathcal{E}} = 0$, since this occurs even in the most interesting cases (skin effect for a flat boundary surface and the like). We note that, as is clear from Eqs. (22), (22a) and everything that has been said, Eq. (17) has a well-known meaning if $\eta = \eta_{el} + \eta_{ee}$ is put in place of η_{ee}. Then we may, of course, speak of "electronic viscosity", having in mind that it is determined by collisions of the electrons with the lattice.

[18] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, (Leipzig-Berlin, 1912)

The order of magnitude of ν'_{el} and ν_{el} is the same. Hence Eq. (30) for $\omega = 0$ gives $\eta_{el} \sim \frac{mn_0v_0}{5} l_l$, which coincides with the estimate made earlier. This same is related also, of course, to the magnitude $\xi_{dl} \sim \sigma_0 l_l^2 / 5$ (it being assumed that $\omega = 0$).

We shall now take account of interelectronic collisions, assuming that $\xi = \xi_{el} + \xi_{ee}$, where $|\xi_{ee}| \ll |\xi_{el}|$. The calculations here are analogous to those which led to Eq. (14). In them we shall not take account of exchange processes, since in the case under consideration the chief term occurring in an analysis in terms of powers of kT/E_0 is conserved even when exchange processes are neglected, so that these processes do not change the order of magnitude of ξ_{ee}. As a result,

$$\frac{\xi_{ee}}{\xi_{el}} = \frac{\eta_{ee}}{\eta_{el}} = -\frac{\nu'_{ee}}{\nu'_{el} + i\omega}, \tag{31}$$

where

$$\nu'_{ee} = q'_{\mathrm{eff}} n_0 v_0 (kT / E_0)^2,$$

$$q'_{\mathrm{eff}} = \frac{9\pi\zeta(3)}{2} \int_{-1}^{+1} dx \frac{q_{ee}(\sqrt{2}\, p_0 \sqrt{1-x}, 0)}{\sqrt{2}\sqrt{1+x}} \times \left(\frac{1}{6} - \frac{2}{3}x + \frac{1}{2}x^2\right).$$

Equation (31) is of the same order as Eq. (15) and, for example, in the statistical case $|\xi_{ee}/\xi_{el}| \sim l_l/l_{ee}$, in agreement with the result of an elementary consideration. Along with this it is not at once evident in the elementary calculation that the magnitude of ξ_{ee} is negative [we integrated this moment above; the fact that $\xi_{ee} < 0$ is clear from a comparison of Eq. (16) with (15), while taking account of the necessity of substituting $\nu'_{el} + \nu'_{ee}$ for ν'_{el} in Eq. (29)] . Thus, taking account of interelectronic interaction leads only to the appearance of a small correction to the coefficients σ' and ξ in Eq. (22a), and it is clear from Part 1 that this correction may be completely disregarded. Thus the effect of the "electronic viscocity" connected with interelectronic collisions is negligibly small, since it is completely masked by the corrections connected with the anomalous character of the skin effect.

3. Relations (22), (22a) and analogous relations are applicable only when the second term is small compared to the first, that is, under the condition $\sigma' \gg \xi/\delta^2$, where δ is a characteristic distance within which the field $\vec{\mathcal{E}}$ changes (usually δ is the skin depth). However, even under this condition Eq. (22) cannot be used to find the correction to the formula obtained in the theory of the normal skin effect. This is due to the fact that relations (22), (22a) are valid only far from the boundaries. When the boundaries are taken into account, there appear, generally speaking, terms of the order l_l/δ (see reference 17) in the expressions for j_t and the surface impedance. It is already clear from this that relations (22), (22a) cannot without further analysis be applied in the theory of the skin effect, no matter what boundary conditions be introduced. Moreover, the boundary condition proposed in reference 1 is manifestly incorrect. Thus it is proposed in reference 1 that at the boundary of a metal $j_t = 0$, whereas from the boundary conditions used in a more general kinetic approach to the problem a completely different result is obtained; to wit, if the reflection of the electrons from the boundary is specular, then, as may readily be shown, $j_t = \sigma'\vec{\mathcal{E}}$ on the boundary for $l_l \to 0$, but for diffuse reflection $j_t = \frac{1}{2}\sigma'\vec{\mathcal{E}}$ on the boundary. In the latter case, which more nearly corresponds to reality, the function $j_t(z)$ for small l_l increases rapidly with distance from the boundary (distance from the boundary is z , with current j_t directed in a perpendicular direction); at a certain distance $z \sim l_l$ $dj_t/dz = 0$, $j_t \approx \sigma'\vec{\mathcal{E}}$, and subsequently j_t begins to fall**.

On the basis of what has been said it is clear that in the theory of the skin effect there are no bases for using Eq. (22), no matter what boundary conditions it fulfills, but that it is expedient to turn directly to the solution of the kinetic equation (see references 16, 17). Here, in particular in the interesting limiting case of the clearly expressed anomalous skin effect, when $l_l \gg \delta$, the results do not in general depend on the collisions of the electrons with the lattice and with each other. In the general case the role of interelectronic collisions may be taken into account by changing the number of collisions ν_{el} to $\nu_{el} + \nu_{ee}$. The corresponding correction is always small, since $\nu_{ee} \ll \nu_{el}$ always. Moreover, this correction in all probability

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** Hence if we prescind from all other remarks, the boundary condition $dj_t / dz = 0$ (see reference 13), corresponding to the condition on the free surface of a viscous liquid, may appear justified to one extent or another. For reasons clear from the text we shall not consider this question in greater detail.

lies beyond the limit of exactness to which existing formulas of the theory of the anomalous skin effect may pretend, especially in the region where $l_l \lesssim \delta$. This becomes clear if we recall that the formulas of which we are speaking (for example, the expression for the surface impedance Z) are obtained [16,17,2] on the basis of assumptions concerning the sphericity of the Fermi surface, the specific character of the reflection of the electrons from the metal boundary and the possibility of approximating the integral of the collisions by means of an expression of the type of Eq. (7) even in the vicinity of the boundary, where the distribution function is not a slowly changing function of the angles in the velocity space. Thus in the present state of the question it is not necessary to take account of interelectronic collisions in the theory of the anomalous skin effect. This refers, in particular, even to the far infrared region of the spectrum‡.

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‡ In view of what has been said, it is clear that the work of reference 1 is incorrect both with respect to the method employed in it and with respect to the results obtained. Moreover, it is necessary to keep in mind that on going over to sufficiently high frequencies, where the condition $\hbar\omega \ll kT$ is not fulfilled, calculations based on the ordinary use of the kinetic equation are already invalid, generally speaking, on account of the necessity of taking account of quantum effects (see reference 19, Sec. 4B). These quantum effects were not taken into account above. The carrying out of a similar calculation using a quantum kinetic equation 20 would be very useful. In the region $\hbar\omega \gtrsim kT$ interelectronic collisions may be essential (see reference 19, Sec. 4B).

[19] V. L. Ginzburg and G. P. Motulevich, Usp. Fiz. Nauk SSSR 55, 469 (1955)

[20] Iu. L. Klimontovich and V. P. Silin, J. Exper. Theoret. Phys. USSR 23, 151 (1952)

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