Multiple Production of Particles in the Collisions of High Energy Nucleons with Nuclei

S. Z. BELEN'KII AND G. A. MILEKHIN
P. N. Lebedev Institute of Physics, Academy of Sciences, USSR
Moscow State University
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The collisions of high energy nucleons with nuclei are examined in a statistical theory of the multiple production of particles. The relation is calculated between the entropy of the nucleon-nucleus system (as determined by the initial stage of the collision), and the number of particles resulting when the system flies apart into individual particles. The entropy of the system is calculated using relativistic hydrodynamics. The dependence of the number of produced particles on the energy of the incident nucleon and on the atomic weight of the nucleus is found.

1. IN the work of Fermi[1] and subsequently Landau[2] a statistical theory was developed for the multiple production of particles in collisions of nucleons of very high energy. Only the collisions of nucleons with nucleons were examined, while the collisions of nucleons with nuclei were not investigated. In experiments, it is precisely collisions with nuclei that are observed, and without a theoretical study (of this subject) it is, strictly speaking, impossible to compare theory and experiment. The aim of this paper is to clarify some problems connected with the interaction of nucleons with nuclei at very high energies.

Let us note at first that Landau's[2] theoretical treatment of nucleon-nucleon collisions can be divided into two parts: In the first part the dependence of the total number of produced particles on the energy of the initial nucleons is calculated. In the second part, the angular and energy distribution of particles is obtained. The calculation of the total number of particles, using general thermodynamic relations, is relatively easy. The second part of the problem requires the complex methods of relativistic hydrodynamics. The results obtained for the total number of particles are much more accurate than for the angular and energy distributions. Because of approximations in the solutions, only the order of magnitude of the latter can be obtained.

In this paper we will limit ourselves to the first part of the nucleon-nucleus problem --- the calculation of the total number of produced particles as a function of the incident nucleon energy and the atomic number of the struck nucleus.

Consider the derivation of the energy dependence of the total number of particles in the nucleon-nucleon problem[2]. The following considerations are used. At the moment of nucleon collision, a system arises in which the mean free path is small compared to the dimensions. Thus the expansion is hydrodynamic in nature; the number of particles in the system remains undetermined during the expansion process, and becomes fixed only at the moment the system flies apart into separate particles.

Since during the whole expansion the motion is adiabatic, the system entropy remains constant until the system flies apart into separate particles. At the moment the system falls apart, the entropy of each small region is proportional to the number of particles in it. Summing over all such regions we have for the entire system:

$$N = \mathrm{const}\cdot S, \tag{1}$$

where N is the total number of particles, and S is the total system entropy. Since the entropy remains constant until the moment of disruption, it is sufficient to calculate it at the beginning of the collision, immediately after the condensation of the system. It follows that in the center of mass frame, all matter is at rest immediately after the collision. Let E' be the energy of the nucleons in the center of mass frame. The total system entropy is proportional to $\epsilon^{3/4}\, V$, where ϵ is the energy density and V the volume in which the energy is distributed (this, if the equation of state for the system is $p = \epsilon/3$). Since V, due to the Lorentz contraction, transforms back proportional to E', and since the energy E in the laboratory system is proportional to E'^2, we finally obtain:

$$S \sim E^{1/4}. \tag{2}$$

Taking account of Eq. (1), we have

$$N \sim E^{1/4}. \tag{3}$$

[1] E. Fermi, Progr. Theor. Phys. **5**, 570 (1950)

[2] L. D. Landau, Izv. Akad. Nauk SSSR **17**, 51 (1953)

Let us now consider the collision of a nucleon with a nucleus. Since the distance between nucleons in the nucleus is of the order of the nuclear force range, it is necessary to examine the process of particle production in the entire nuclear volume traversed by the incident nucleon. It is not difficult to see that both assumptions used to derive Eq. (3) need modification in the nucleon-nucleus case. In the first place, due to the presence of the nucleons participating in the process it is impossible to consider the number of particles proportional to the entropy S. Secondly, the calculation of the entropy change must be modified, since the asymmetry of the collision makes it impossible to indicate a time after the collision when all matter is at rest.

2. We begin by finding the dependence of the number of particles produced on the entropy. Let the disruption of the system into single particles take place at a temperature T_k, of the order $m_\pi c^2$, where m_π is the mass of a π-meson, and c the velocity of light. Particles at this temperature can be considered to form a perfect gas. Since the temperature at disruption is unknown, it may be a relativistic gas. We shall present expressions for the number of particles and for the entropy of a Bose gas (π-mesons) and a Fermi gas (nucleons). We shall not consider other particles, since their contribution is relatively small.

For the density of nucleons, we have

$$n_n = \left(\frac{kT}{\hbar c}\right)^3 \frac{g_n}{2\pi^2} [F_1(z_1, y_1) + F_1(z_1, y_2)], \quad (4)$$

where

$$z_1 = \frac{Mc^2}{kT}, \qquad y_1 = \frac{\mu_1}{kT}, \qquad y_2 = \frac{\mu_2}{kT}.$$

Here g_H is the statistical weight, in this case four (two spin states and two charge states), M the nucleon mass, μ_1 and μ_2 chemical potentials, respectively, of nucleons and antinucleons [the second term of (4) is for anti-nucleons].

The density of π-mesons is

$$n_\pi = \left(\frac{kT}{\hbar c}\right)^3 \frac{g_\pi}{2\pi^2} F_2(z,0). \quad (5)$$

Here $\quad g_\pi = 3, \quad z = m_\pi c^2 / kT,$

$$F_{1,2}(z,y) = z^3 \int_0^\infty \frac{\exp\{-z\sqrt{1+x^2}+y\}}{1 \pm \exp\{-z\sqrt{1+x^2}+y\}} x^2 dx. \quad (6)$$

The plus sign (corresponding to the function F_1) is for fermions, the minus sign (corresponding to the function F_2) for bosons[3].

[3] S. Z. Belen'kii, Dokl. Akad. Nauk SSSR **99**, 523 (1955)

If $z > y$, then $F_{1,2}(z, y)$ may be put in series form:

$$F_{1,2}(z, y) = z^2 \sum_{m=0}^{\infty} (\mp 1)^m \frac{\exp\{y(m+1)\} K_2[z(m+1)]}{(m+1)}, \quad (7)$$

where $K_2(z)$ is a modified Bessel function of the second kind.

We now derive an expression for the particle entropy. For the density of entropy of nucleons and anti-nucleons, following from the relation $S = (E - N_\mu - \Omega)/T$, we have

$$S_n = k\left(\frac{kT}{\hbar c}\right)^3 \frac{g_n}{2\pi^2} [G_1(z_1, y_1) + G_1(z_1, y_2) - y_1 F_1(z_1, y_1) - y_2 F_1(z_1, y_2)]. \quad (8)$$

For the density of meson entropy we obtain

$$S_\pi = k\left(\frac{kT}{\hbar c}\right)^3 \frac{g_\pi}{2\pi^2} G_2(z, 0), \quad (9)$$

where

$$G_{1,2}(z, y) = \Phi_{1,2}(z, y) - \Psi_{1,2}(z, y) \quad (10)$$

$$\Phi_{1,2}(z, y) = z^4 \int_0^\infty \frac{x^2 \sqrt{1+x^2} dx}{\exp\{-y + z\sqrt{1+x^2}\} \pm 1}$$

$$\Psi_{1,2}(z, y) = \mp z^3 \int_0^\infty \ln(1 \pm \exp\{y - z\sqrt{1+x^2}\}) x^2 dx.$$

The functions $\Phi_{1,2}(z, y)$ [3] and $\Psi_{1,2}(z, y)$ are related to the energy and thermodynamic potential Ω of the particles by the following:

$$\varepsilon = \frac{E}{V} = kT \frac{g}{2\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \Phi_{1,2}(z, y); \quad (11)$$

$$\omega = \frac{\Omega}{V} = kT \frac{g}{2\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \Psi_{1,2}(z, y),$$

where ϵ and ω are the energy density and thermodynamic potential density respectively.

For the case $z < y$, $G_{1,2}(z, y)$ may be written as the following series:

$$G_{1,2}(z, y) = z^2 \sum_{m=0}^{\infty} e^{y(m+1)} (\mp 1)^m \times \frac{4K_2[z(1+m)] + z(1+m) K_1[z(1+m)]}{(1+m)^2}, \quad (12)$$

$z = \frac{m_\pi c^2}{kT}$ (m_π=mass of the π-meson)	$F_1(z, 0)$	$F_2(z, 0)$	$\Phi_1(z, 0)$	$\Phi_2(z, 0)$	$G_1(z, 0)$	$G_2(z, 0)$	$\alpha(z)$
0	1,803	2.40	5,68	6.49	7,57	8,65	0,25
0,5	1.72	2.17	5,58	6,30	7,37	8,31	0,213
0,7	1.65	2.02	5,47	6,12	7,19	8.02	0,216
0.9	1.56	1.86	5.33	5,90	6,95	7,67	0.221
1	1.52	1,78	5.24	5,78	6,81	7,48	0,223
1.2	1.41	1.62	5.05	5,51	6,51	7,07	0,222
1.5	1,25	1.39	4,72	5.06	6,00	6,42	0.215
2	0,982	1,05	4,07	4,27	5,07	5,31	0,198
3	0.546	0.561	2,72	2.78	3,27	3,33	0.169
6	0,0559	0,0599	0,471	0,471	0.531	0,531	0.113
7	0.0268	0.0268	0,237	0.237	0,263	0.263	0.102
8	0.0117	0.0117	0.115	0.115	0,127	0.127	0,0927

where $K_1(z)$ is a modified Bessel function of the first kind.

The functions $F_{1,2}(z, 0)$, $\Phi_{1,2}(z, 0)$, and $G_{1,2}(z, 0)$ which determine the density of particles, energy and entropy for fermions and bosons are given in the Table.

Since nucleons and anti-nucleons annihilate in pairs, creating various particles with total chemical potential of zero, it follows that the chemical potential of an anti-nucleon is equal and opposite to that of a nucleon. That is, $y_1 = y$ and $y_2 = -y$.

From Eqs. (4)-(9) we obtain the following expression for the total entropy of the system:

$$\frac{S}{k} = \frac{G_1(z_1, y) + G_1(z_1, -y) - y\,[F_1(z_1, y) - F_1(z_1, -y)}{F_1(z_1, y) + F_1(z_1, -y)} N_{\text{H}} + \frac{G_2(z, 0)}{F_2(z, 0)} N_\pi. \tag{13}$$

Here N_{n} is the total number of nucleons and anti-nucleons, and N_π the total number of π-mesons in the system. We have made the transition from particle density to the total number with the assumption that each region of the system at disruption has the same average values of z and y. Furthermore, we take into account the conservation of nuclear charge, which implies that the difference between the number of nucleons N_{nn} and anti-nucleons N_{an} must be equal to the number of initial nucleons N_0

$$N_{\text{nn}} - N_{\text{an}} = \left(\frac{kT}{\hbar c}\right)^3 \frac{g_{\text{H}}}{2\pi^2} [F_1(z_1, y) - F_1(z_1, -y)]\, V = N_0. \tag{14}$$

Here V is the total volume of the system.

Let us first examine the case $N_0 = 0$. It follows from Eq. (14) that $y = -y = 0$. From Eq. (13) we obtain

$$N^* = \alpha \frac{S}{k},$$

where

$$\alpha(z) = \frac{2g_{\text{n}} F_1(z_1, 0) + g_\pi F_2(z, 0)}{2g_{\text{n}} G_1(z_1, 0) + g_\pi G_2(z, 0)} \tag{15}$$

and $N^* = N_m + N_\pi$, i.e., N^* is equal to the sum of particles produced with $N_0 = 0$. The function $\alpha(z)$ is given in Table I. It is evident from the Table that $\alpha(z)$ is weakly dependent on z, i.e., on the temperature.

It is not difficult to show that when $kT \ll m_\pi c^2$, $\alpha(z) \approx 1/(z + 5/2)$, i.e., $N^* \approx [1/(z + 5/2)] \times (S/k)$.

Let us now return to Eq. (13). Instead of S we use the ratio of the number of particles N^* (produced with $N = 0$) to N_0. Multiplying both sides of Eq. (13) by $\alpha(z)$ and dividing by N_0, we obtain

$$\frac{N^*}{N_0} = \left[\frac{G_1(z_1, y) + G_1(z_1, -y)}{F_1(z_1, y) - F_1(z_1, -y)} - y + \frac{g_\pi}{g_{\text{n}}} \frac{G_2(z, 0)}{F_1(z_1, y) - F_1(z_1, -y)}\right] \alpha(z). \tag{16}$$

On the other hand, it is not difficult to see that the total number of particles is

$$\frac{N}{N_0} = \frac{N_\pi + N_{\text{n}}}{N_0} = \frac{g_\pi F_2(z, 0) + g_{\text{n}}[F_1(z_1, y) + F_1(z_1, -y)]}{g_{\text{n}}[F_1(z_1, y) - F_1(z_1, -y)]}. \tag{17}$$

The two equations (16) and (17), together with the parameter y, determine the dependence of N/N_0 on N^*/N_0 (i.e., on the ratio of entropy to the initial number of nucleons). If we assume $z_1 \gg y$, and therefore leave only the first terms of the $F_1(z, y)$ and $G_1(z, y)$ series expansions, the solution of the equations becomes materially easier. Physically this assumption means that we neglect the difference between a Fermi and Maxwell distribution for the nucleons, which is allowable in this case. Since the quantity y then appears as an exponential in the distribution function, it is not difficult to show that

$$F_1(z_1, y) - F_1(z_1, -y) = 2F_1(z_1, 0) \operatorname{sh} y,$$

$$F_1(z_1, y) + F_1(z_1, -y) = 2F_1(z_1, 0) \operatorname{ch} y,$$

$$G_1(z_1, y) + G_1(z_1, -y) = 2G_1(z_1, 0) \operatorname{ch} y.$$

As a result, instead of Eqs. (16) and (17), we have

$$\frac{N^*}{N_0} = \left[\frac{G_1(z_1, 0)}{F_1(z_1, 0)} \operatorname{cth} y - y + \frac{g_\pi G_2(z, 0)}{2g_n F_1(z_1, 0) \operatorname{sh} y}\right] \alpha(z), \qquad (18)$$

$$\frac{N}{N_0} = \frac{N_\pi}{N_0} + \operatorname{cth} y,$$

$$y = \operatorname{Arsh}\left[\frac{g_\pi F_2(z, 0)}{2g_n F_1(z_1, 0)} \frac{N_0}{N_\pi}\right].$$

From these equations the ratio N/N_0 as a function of N^*/N_0 was found. This is shown in Fig. 1.

If the number of initial nucleons is sufficiently large, and if the critical temperature T_k is not too high, then $\sinh y = \frac{g_\pi F_2(z)}{2g_n F_1(z)} \frac{N_0}{N_\pi}$ is much larger than unity. This condition means that the number of initial nucleons significantly surpasses the number of nucleons produced with $N_0 = 0$.

If sinh y and $y > 1$, then Eq. (18) becomes

$$\gamma = \delta + \alpha(z) \ln(\delta - 1) + B(z), \qquad (19)$$

where

$$\gamma = \frac{N^*}{N_0}, \qquad \delta = \frac{N}{N_0},$$

$$B(z) = \alpha(z)\left[\frac{G_1(z_1, 0)}{F_1(z_1, 0)} - \ln\left(\frac{g_\pi F_2(z, 0)}{g_n F_1(z_1, 0)}\right)\right] - 1.$$

With $z \gg 1$

$$B(z) = \frac{\ln(g_n/g_\pi)(M/m)^{3/2}}{z + 5/2}, \qquad z = \frac{m_\pi c^2}{kT}.$$

If $\gamma > 1$, then $\gamma \approx \delta$; with $\delta - 1 < 1$

$$\delta - 1 = \exp\left\{-\frac{B(z) + 1 - \gamma}{\alpha(z)}\right\};$$

if also $z \gg 1$, then

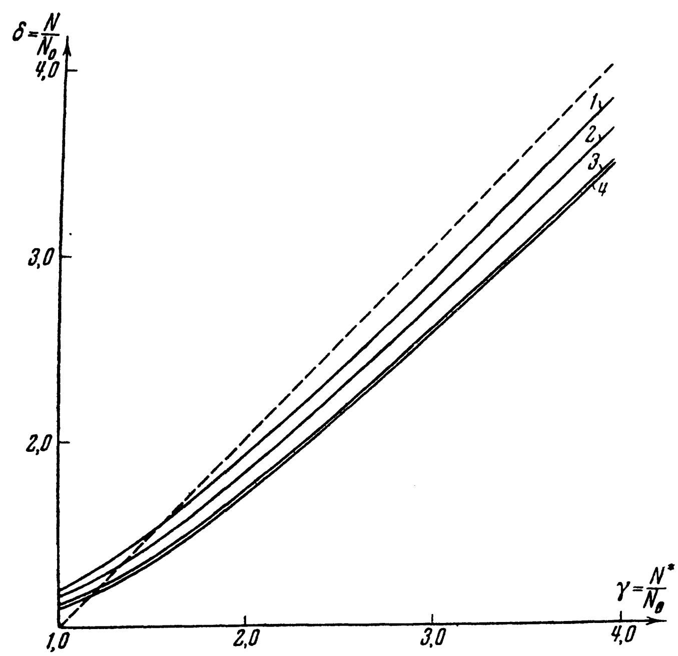

FIG. 1. kT/mc^2 is: *1.* 1.25; *2.* 1.0; *3.* 0.67; *4.* 0.5. The dotted line corresponds to $\gamma=\delta$.

$$\frac{N-N_0}{N_0}=\delta-1\simeq\frac{g_n}{g_\pi}\left(\frac{M}{m}\right)^{3/2}$$
$$\times\ \exp\left\{-\left(\frac{mc^2}{kT}+\frac{5}{2}\right)\left(1-\frac{N^*}{N_0}\right)\right\}.$$

Figure 1 [calculated from Eq. (18)] shows, for the various chosen temperatures of disruption T_k, that N/N_0 is always less than N^*/N_0 until the $\gamma \approx 2$ region. (Note that N includes the number of initial nucleons N_0.) The difference between N/N_0 and N^*/N_0 is larger, the lower the temperature T_k. Already at $N^*/N_0 = 3$ and $z = mc^2/kT = 2$, $N/N_0 = 2.54$, i.e., the ratio $N/N^* = 0.85$. For $\gamma < 2$, the number of newly produced particles falls rapidly, and approaches unity. It is seen from this that the relation

$$N = \text{const} \cdot S$$

is also valid for the collision of a nucleon with a nucleus until N^*/N_0 is of the order of two. N is the sum of particles produced during the collision and the initial nucleons.

3. Let us now calculate the change in entropy. In the nucleon nucleon problem it was not necessary to examine the mechanism of compression, since the results could be obtained immediately from symmetry considerations. The situation is quite different for the nucleon-nucleus collision. Let us apply our model of nuclear matter as a continuous medium to the first stage of the collision --- the compression.

It will turn out from the following that the most convenient system of coordinates is that in which the nucleon and nucleus have equal and oppositely directed velocities. In this reference frame, due to the Lorentz contraction, both nucleon and nucleus look loke very thin disks. Thus, the problem can be considered one dimensional. In this case the collision of the nucleon with the nucleus appears as the collision of the nucleon with a tube cut from the nucleus[4], with a cross section equal to that of the nucleon and a length between the contracted diameters of nucleus and nucleon. With the close approach of the nucleon to the tube, impact waves propagate in both directions with velocity D through the nuclear and nucleon matter (Fig. 2). Of course, we can speak of the propagation of impact waves in the nucleon only provisionally. Hydrodynamic considerations are used in this case for orientation.

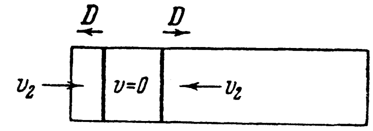

FIG. 2

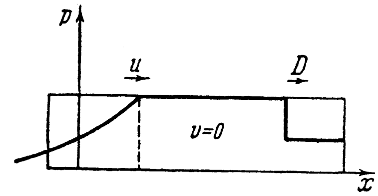

FIG. 3. The dotted line is the boundary of the rarefaction wave.

Because of the equality of velocities v_2 of the approaching particles in the chosen reference frame, the matter between the impact waves is at rest. The impact wave traveling to the left will reach the edge sooner than the wave traveling to the right (in contrast to the nucleon-nucleon situation where both waves reach an edge at the same time). When the impact wave reaches the left edge, the compression ceases, and an outflow of matter begins (Fig. 3). This means that a wave of refraction travels to the right with a velocity equal to that of sound in the medium. At the same time, the impact wave continues traveling to the right, since it has not yet reached the right edge. The calculation of the entropy will be different depending on whether the rarefaction wave overtakes the impact wave before it reaches the right edge. It is known that the speed of sound in a medium (equation of state $p = \epsilon/3$) is[5]

$$u = c/\sqrt{3}. \tag{20}$$

Let us now calculate the velocity of the impact wave. We change to a coordinate system in which the impact wave is stationary. Then (Fig. 2) the velocity of matter behind the impact wave is D, while ahead of the wave it is

$$v_2' = \frac{v_2 + D}{1+(v_2D/c^2)}.$$

Because of the continuity of energy and momentum flows through the impact wave front, we have[5,6]

$$\frac{p_1+(D^2/c^2)\,\varepsilon_1}{1-(D^2/c^2)}=\frac{p_2+(v_2'^2/c^2)\,\varepsilon_2}{1-(v_2'^2/c^2)}, \tag{21}$$

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[4] I.L. Rozental' and D.S. Chernavskii, Usp. Fiz. Nauk 52, 185 (1954)

[5] L. D. Landau and E. M. Lifshitz, *Mechanics of Continuous Media*, 1954

$$\frac{(D/c)(p_1+\varepsilon_1)}{1-(D^2/c^2)} = \frac{(v_2'/c)(p_2+\varepsilon_2)}{1-(v_2'^2/c^2)}. \quad (22)$$

Here p_1 and ϵ_1 are the pressure and energy density behind the impact wave, and p_2 and ϵ_2 the pressure and energy density ahead of the wave. Dividing the first equation by the second, we obtain

$$\frac{p_1+(D^2/c^2)\varepsilon_1}{(D/c)(p_1+\varepsilon_1)} = \frac{p_2+(v_2'^2/c^2)\varepsilon_2}{(v_2'/c)(p_2+\varepsilon_2)}. \quad (23)$$

Since the velocity v_2 of the colliding particles is very close to that of light, the velocity $v_2^{1\prime}$ will be also. Assuming $v_2^{1\prime} = c$ we see that the right side of Eq. (23) is unity, and consequenlty our results are independent of the equation of state for the matter ahead of the impact wave.

Making use of the equation of state for the matter behind the impact wave ($p_2 = \epsilon_2/3$), we obtain the following for D:

$$\left(\frac{1}{3}+\frac{D^2}{c^2}\right)\Big/\frac{4}{3}\frac{D}{c} = 1,$$

from which it follows that

$$D = {}^1/_3\, c. \quad (24)$$

It is now possible to find the minimum tube length l_k in which the rarefaction wave will overtake the impact wave. This is determined by the ratio

$$\frac{(l_k/d)-1}{D} = \frac{(l_k/d)+1}{u} \quad \text{or} \quad \frac{l_k}{d} = \frac{D+u}{u-D},$$

where d is the nucleon "diameter".

Substituting the values of u and D, we obtain

$$l_k/d = 3.7. \quad (25)$$

Let us assume that the tube length l is smaller than l_k, i.e., that the impact wave traveling to the right reaches the edge before it is overtaken by the rarefaction wave. In this case it is very easy to calculate the entropy. One calculates the entropy of the separate regions of the system immediately after the passage of the impact wave, at which time they are at rest in our reference frame. It is not difficult to see that the change in entropy of the entire system is then

$$\frac{S}{S_0} = \frac{1}{2}\left(\frac{l}{d}+1\right) \quad \text{or} \quad \frac{l}{d} \leqslant 3.7, \quad (26)$$

where S_0 is the change of entropy in the nucleon-nucleon process, and l the tube length. This result may be obtained with the aid of a calculation applicable to the nucleon-nucleon problem (see above), keeping in mind that the effective volume is that of the tube plus that of the nucleon, and that the matter is at rest in the frame where velocities of the initial particles are equal and opposite. In an outline of such a calculation[4] the assumption is made that matter is at rest in the center of mass frame, but this is incorrect.

For tube lengths exceeding $l_k = 3.7$ the solution becomes more complex. The rarefaction wave overtakes the impact wave in this case, but cannot cross its wave front, since the impact wave travels in the matter of the oncoming nucleus with a velocity greater than that of sound in the matter. The rarefaction wave is reflected by the impact wave. A region is formed bounded by the impact wave on the right, and by the rarefaction wave on the left (Fig. 4).

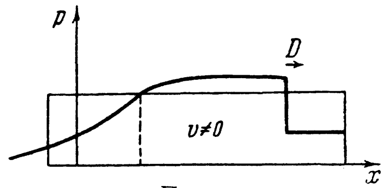

FIG. 4

To describe the motion of the medium in this region, we refer to the arbitrarily similar motion of a compressible gas[5]. In the ultra-relativistic case the equation of motion becomes[6]

$$3\frac{\partial^2\chi}{\partial\eta} - \frac{\partial^2\chi}{\partial y^2} - 2\frac{\partial\chi}{\partial y} = 0. \quad (27)^*$$

Independent variables here are the quantities $\eta = \operatorname{arctanh} v/c$, where v is the velocity of the medium, c the velocity of light, and $y = \ln(T/T_0)$, T the temperature of the medium, T_0 the temperature at points where $v = 0$. The coordinate x and time t are expressed as a function of χ by

$$x = e^{-y}\left(\frac{\partial\chi}{\partial y}\,\mathrm{sh}\,\eta - \frac{\partial\chi}{\partial\eta}\,\mathrm{ch}\eta\right), \quad (28)$$

$$t = e^{-y}\left(\frac{\partial\chi}{\partial y}\,\mathrm{ch}\,\eta - \frac{\partial\chi}{\partial\eta}\,\mathrm{sh}\eta\right).$$

Thus if the function $\chi(\eta, y)$ is found, Eq. (28) gives x and t as functions of T and v.

The region of interest is bounded on one side by

* In reference 6, the coefficients are different because of typographical errors.

[6] I. M. Khalatnikov, J. Exper. Theoret. Phys. USSR 26, 529 (1954)

the rarefaction wave, and on the other by the impact wave. Let us determine the boundary conditions for the function χ. It follows from reference 6 that, at the rarefaction wave boundary,

$$\chi = 0 \quad \text{for} \quad \eta = \sqrt{3}\, y. \tag{29}$$

must be satisfied.

Let us consider the boundary condition at the impact wave. Since in this case the matter behind the impact wave is not at rest, transformation to a reference frame in which the wave front is stationary gives for the velocity behind the impact wave:

$$v_1 = \frac{v_1 + D}{1 + (v_1 D / c^2)}. \tag{30}$$

Here v_1 is the velocity behind the impact wave in the reference frame where the incident particle velocities are equal.

Equations (21) - (24) remain applicable to the present case if $v_1{}'$ is substituted for D. It follows that $v_1{}' = 1/3\, c$ [see Eq. (24)]. Equation(30) then gives a relation between v_1 and D which may be written

$$D = \frac{dx}{dt} = \frac{1 + 3\operatorname{th}\eta}{3 + \operatorname{th}\eta}, \tag{31}$$

since $v_1 = -\tanh\eta$. Using Eqs. (21) - (24) with v_1 substituted for D, it is not difficult to obtain the following expression which is obeyed on the impact wave:

$$\frac{\varepsilon}{\varepsilon_0} = \left(\frac{T}{T_0}\right)^4 = \frac{1 - (v_1 / c)}{1 + (v_1 / c)}.$$

In the variables η and y

$$\left(v_1 = -\operatorname{th}\eta, \quad y = \ln\frac{T}{T_0}\right);$$

this means

$$\eta = 2y. \tag{32}$$

Substituting the value of dx/dt calculated from Eq. (28) into Eq. (31), and using Eq. (32), we obtain the following condition on the impact wave:

$$\left(3\frac{\partial}{\partial y} + 5\frac{\partial}{\partial\eta}\right)\left(1 - \frac{\partial}{\partial y}\right)\chi = 0 \quad \text{for} \quad \eta = 2y. \tag{33}$$

Let us change to the variables

$$\alpha = \eta - 2y, \quad \beta = \sqrt{3}\, y - \eta. \tag{34}$$

In the new variables the equations for χ and the boundary conditions take the following form:

$$\frac{\partial^2\chi}{\partial\alpha^2} - 2\sqrt{3}(2 - \sqrt{3})\frac{\partial^2\chi}{\partial\alpha\,\partial\beta} - 4\frac{\partial\chi}{\partial\alpha} + 2\sqrt{3}\frac{\partial\chi}{\partial\beta} = 0; \tag{35}$$

$$\chi = 0 \text{ when } \beta = 0,$$

$$\left[\frac{\partial}{\partial\alpha} + (5 - 3\sqrt{3})\frac{\partial}{\partial\beta}\right] \times \left[1 + 2\frac{\partial}{\partial\alpha} - \sqrt{3}\frac{\partial}{\partial\beta}\right]\chi = 0 \quad \text{when} \quad \alpha = 0. \tag{36}$$

Let us apply a Laplace transformation to the variable β

$$f(\alpha, p) = \int_0^\infty \chi(\alpha, \beta)\, e^{-p\beta}\, d\beta \tag{37}$$

and search for a solution of $f(\alpha, p)$ in the form $a(p)\, e^{k\alpha}$. Then Eqs. (35) and (36) yield the following algebraic equations for $a(p)$ and k:

$$k^2 - 2\sqrt{3}(2 - \sqrt{3})\, kp - 4k + 2\sqrt{3}\, p = 0, \tag{38}$$

$$[k + (5 - 3\sqrt{3})\, p]\,[1 + 2k - \sqrt{3}\, p]\, a = (9 - 5\sqrt{3})\lambda$$

Here λ is the coordinate value at which the rarefaction wave overtakes the impact wave;

$$\lambda = \left(\frac{\partial\chi}{\partial\beta}\right)_{\substack{\alpha=0\\ \beta=0}} = -\left(\frac{\partial\chi}{\partial\eta}\right)_{\substack{\eta=0\\ y=0}}. \tag{39}$$

Having determined $a(p)$ and k, an inverse Laplace transform would give χ, and thus solve the problem. However, we are not interested in the complete solution, but rather in the change of entropy during the time that the impact wave passes. The entropy change can be written:

$$S_1 = \sigma_0 \int_{t_1'}^{t_2'} s u_1\, dt'. \tag{40}$$

Here σ_0 is the tube cross section, and s is the entropy density after the impact wave; $u_1 = \frac{(v_1'/c)}{\sqrt{1 - (v_1'/c)^2}}$, where $v_1{}'$ is the matter velocity behind the impact wave front (since $v_1{}'/c = 1/3$, $u_1 = 1/2\sqrt{2}$), $t_1{}'$ is the time at which the rarefaction wave overtakes the impact wave, t_2' the time at which the impact wave reaches the right edge and dt' an element of time in the reference frame where the impact wave front is at rest:

$$dt' = dt\sqrt{1-D^2} = \sqrt{dt^2 - dx^2}.$$

Using Eq. (28), the condition $\eta = 2y$ at the impact wave, and the fact that the ratio of entropy densities s/s_0 behaves as $(T/T_0)^3$ when the matter after the impact wave comes to rest, we obtain

$$S_1 = \sigma_0 \frac{1}{9}\int_0^{y_k} e^{2y}\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} - 1\right)\chi\right]_{\eta=2y} dy. \quad (41)$$

Here y_k is the value of y at the instant that the impact wave reaches the edge of the system.

Changing to the variables α and β we have

$$S_1 = \sigma_0 \frac{1}{9(2-\sqrt{3})}\int_0^{\beta_k} e^{-2\beta/(2-\sqrt{3})} \quad (42)$$

$$\times \left[\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}\right)\left(1 + 2\frac{\partial}{\partial\alpha} - \sqrt{3}\frac{\partial}{\partial\beta}\right)\chi\right]_{\alpha=0} d\beta.$$

Multiply both sides by $\exp\left\{\frac{2\beta_k}{2-\sqrt{3}}\right\}$. Let us then carry out a Laplace transformation. Using the curl theorem, we obtain for a representation of the quantity $S_1 \exp\left\{\frac{2\beta_k}{2-\sqrt{3}}\right\}$:

$$\Psi(p) = \int_0^{\infty} S_1 \exp\left\{\frac{2\beta_k}{2-\sqrt{3}}\right\} e^{-p\beta} d\beta$$

$$= \frac{\sigma_0}{9(2-\sqrt{3})}\frac{1}{p - 2/(2-\sqrt{3})}$$

$$\times \{(k-p)(1+2k-\sqrt{3}p)a - \sqrt{3}\lambda\}.$$

Substitute into this the values of k and a calculated from Eq. (38). (For k a quadratic equation is obtained; the solution with the minus sign before the square root is chosen.) As a result, we have

$$\Psi(p) = \sigma_0\left[\frac{(5+3\sqrt{3})\lambda}{2\sqrt{3}(q-2)(q-5)}\right. \quad (43)$$

$$\left. + \frac{(\sqrt{3}-1)\lambda}{2\sqrt{3}(q-2)(q-5)}\frac{1}{\sqrt{3}(q-1) + \sqrt{3(q-1)^2+1}}\right],$$

where $q = (2-\sqrt{3})p$.

Inverting the Laplace transform, we have

$$S_1 \exp\{2\beta_k/(2-\sqrt{3})\} \quad (44)$$

$$= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \Psi(p)\, e^{p\beta}\, dp.$$

The integral in Eq. (44) is taken in the complex plane along a line which is to the right of all poles of the integrand.

Substituting Eqs. (43) into (44), the integration is carried out. We make use of the known relation[7]

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{dp\, e^{tp}}{p+\sqrt{p^2+1}} = \frac{J_1(t)}{t}.$$

As a result we obtain:

$$S_1 = \sigma_0\left[\frac{5+3\sqrt{3}}{6\sqrt{3}} s_0\lambda(e^{3\beta_k'} - 1)\right. \quad (45)$$

$$\left. + \frac{\sqrt{3}-1}{6\sqrt{3}} s_0\lambda e^{-3\beta_k'}\int_0^{\beta_k'}\left(e^{4z} - e^{z}\right)\frac{J_1((\beta_k'-z)/\sqrt{3})}{\beta_k'-z} dz\right],$$

where $\beta_k' = \beta_k/(2-\sqrt{3})$.

To determine β_k we have the following relation:

$$x_k + t_k - t_0 = L, \quad (46)$$

where t_0 is the instant at which the rarefaction wave overtakes the impact wave, L the coordinate value of the right edge at this instant, x_k and t_k the coordinate and time when the impact wave reaches the edge of the tube. Expressing x and t through a potential, and remembering that $t_0 = \lambda\sqrt{3}$, we obtain

$$\frac{\partial\chi}{\partial y} - \frac{\partial\chi}{\partial\eta} = (L + \sqrt{3}\lambda)e^{-y_k} \quad (47)$$

or in the variables α and β:

$$-3\left[\frac{\partial\chi}{\partial\alpha} - (\sqrt{3}-1)\frac{\partial\chi}{\partial\beta_k'}\right]_{\alpha=0} \quad (47')$$

$$+ 8\left(\frac{\partial\chi}{\partial\beta_k'}\right)_{\alpha=0} = (L+\sqrt{3}\lambda)e^{\beta_k'}.$$

A similar calculation, carried to higher order, leads to the following expression:

$$(7+4\sqrt{3})e^{4\beta_k'} - 1 \quad (48)$$

[7] V. A. Ditkin and I. I. Kuznetsov, *Handbook of Operational Calculus*, Moscow, 1951

$$+\int_0^{\beta_k}(e^{4z}-1)\frac{J_1[(\beta_k'-z)/\sqrt{3}]}{\beta_k'-z}dz=\frac{6(L+\sqrt{3}\lambda)}{\sqrt{3}(\sqrt{3}-1)\lambda}.$$

The terms containing integrals in Eqs. (45) and (48) contribute not more than 2% ot the total, and may be thus dropped. The coefficient of the first term in Eq. (45) $(5+3\sqrt{3})/6\sqrt{3}$ can be replaced by one to a good degree of approximation. As a result we have

$$S_1=\sigma_0\lambda s_0(e^{3\beta_k'}-1), \quad (49)$$

$$(7+4\sqrt{3})e^{4\beta_k'} = \frac{\sqrt{3}(\sqrt{3}+1)}{\lambda}(L+\sqrt{3}\lambda)+1.$$

It is not difficult to be convinced that $L+\sqrt{3}\lambda = 4l-2d$. Since $\lambda=(3+\sqrt{3})d$, the second of Eq. (49) may be written:

$$(7+4\sqrt{3})e^{4\beta_k'}=4\frac{l}{d}-1.$$

Finally, we have for the change in entropy the following expression :

$$\frac{S}{S_0}=0{,}92\left(\frac{l}{d}-\frac{1}{4}\right)^{3/4} \quad \text{with} \quad \frac{l}{d}\geqslant 3.7. \quad (50)$$

where S_0 is the change in entropy due to the collision of the nucleon with the nucleus, S is the whole change of entropy due to the collision of the nucleon with a tube of length l.

In head on collisions of a nucleon and nucleus $l/d=A^{1/3}$, where A is the atomic number. The value $l=3.7$ corresponds to $A=51$. If Eqs. (26) and (50) are averaged over all possible collisions in the nucleus, from head on, to collisions of the incident nucleon with peripheral nucleons, we obtain (not separately considering lateral nucleon-nucleon collisions, which, we feel, are already accounted for in the equation for nucleon-nucleon collisions):

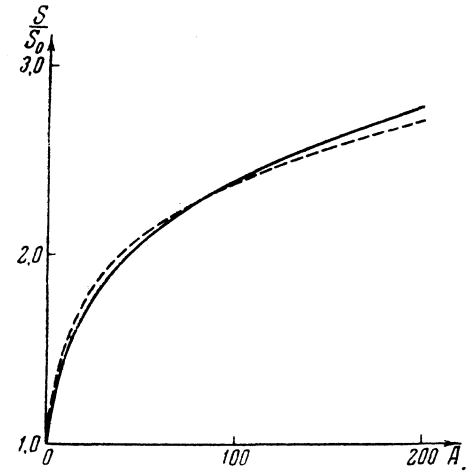

FIG. 5. The dotted line is $S/S_0=A^{0.19}$.

For $A<51$

$$\frac{S}{S_0}=\frac{1}{3}\frac{A-(2A^{1/3}-1)^{3/2}}{(A^{1/3}-1)^2}+0.5; \quad (51)$$

For $A>51$

$$\frac{S}{S_0}=\frac{4}{(A^{1/3}-1)^2}\left[0.167(A^{11/12}-A_0^{11/12}) + \frac{1}{12}(A_0-(2A^{1/3}-1)^{3/2})\right] \quad (52)$$

$$-0.6\frac{A^{2/3}-A_0^{2/3}}{(A^{1/3}-1)^2}+0.5, \text{ where } A_0=51.$$

Figure 5 shows the dependence of S/S_0 on A. The dependence may be approximated with an accuracy of 4% by the expression

$$S/S_0=A^{0.19}. \quad (53)$$

Translated by G. L. Gerstein
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