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# Relativistically Invariant Equations for the Electron which Take the Place of Dirac's System of Equations

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Making use of the properties of real spinors, additional terms are found which make the equations for the electron used in non-relativistic quantum mechanics relativistically invariant. The second order differential equations found in this way, which connect the components of one real spinor, are simpler than those which are obtained through Dirac's theory. The relativistically invariant equations introduced are solved for the hydrogen atom in the absence of external fields. It turns out that in that case the usual fine structure formula is obtained, just as in the Dirac theory. It is shown that the considered second order differential equations can be obtained from a system of relativistically invariant equations of first order which, however, no longer contain one, but two, real spinors.

## 1. INTRODUCTION

A descriptive interpretation of the ideas and concepts used in non-relativistic quantum mechanics for study of the behavior of particles with spin 1/2 was given in reference 1. The problem of finding relativistically invariant equations for the electron must also be reexamined from this point of view.

Currently, as is well known, Dirac's system of equations[2] is considered as the system of relativistically invariant equations which characterize the behavior of an electron in various external fields (see also references 3-6 etc.). This choice was based on the possibility of getting the ordinary non-relativistic equations from them in a special case and also on their ability to explain more exactly the available experimental facts. Most important of such facts is the existence of fine structure in the spectrum of the hydrogen atom, and its explanation by the use of Dirac's equations is accordingly the chief argument in favor of the validity of the Dirac theory. It is obvious that any relativistically invariant equations for the electron which claim to replace Dirac's system of equations must also reduce to the equations of nonrelativistic quantum mechanics in a special case and at the same time explain experimental facts not explained by the latter. Above all, moreover, they must lead to a formula for the fine structure experimentally confirmed.

Relativistically invariant equations for the electron, which satisfy the formulated requirements and which at the same time are simpler than the Dirac equations, will be obtained in the present paper. During the derivation of these equations we will make any hypotheses but will not be guided only by the fact that the equations of non-relativistic quantum mechanics for the electron interpreted with the aid of the graphical concepts developed in reference 1 must be generalized in such a way that they become relativistically invariant, i.e., that they don't change under arbitrary transformations of the general Lorentz group. We will need the mathematical apparatus developed in references 7 and 8.

## 2. DERIVATION OF THE RELATIVISTICALLY INVARIANT EQUATIONS TO REPLACE DIRAC'S SYSTEM OF EQUATIONS

As is known, the components of the momentum operator and the energy operator which are used in

[1] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 653 (1953)

[2] P. A. M. Dirac, Proc. Roy. Soc. London 117, 610; 118, 351 (1928)

[3] V. A. Fock, *Principles of Quantum Mechanics*, Kubuch, 1932

[4] Louis De Broglie, *The Magnetic Electron*, Paris, 1934

[5] W. Pauli, *General Principles of Wave Mechanics*, (Ann Arbor, Mich) Edwards 1946

[6] P. A. M. Dirac, *Quantum Mechanics*, (1935) 2nd Ed. (Oxford)

[7] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 667 (1953)

[8] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **28**, 524; Soviet Phys. **1**, 411 (1955)

non-relativistic quantum mechanics, from the point of view of graphical representations of the states of particles with spin 1/2, must be written in the form:

$$\hat{p}_k = -\hbar I \frac{\partial}{\partial x^k}, \quad \hat{E} = \hbar I \frac{\partial}{\partial t} \tag{1}$$

(see reference 1). Therefore, on going over to four-dimensional pseudo-Euclidian space, we shall have the following relations for a real spinor which is a "proper" spinor of these operators:

$$-\hbar I \frac{\partial \psi}{\partial x^\alpha} = p_\alpha \psi \tag{2}$$

$$(x^4 = ct,\ p_4 = -E/c,\ \alpha = 1, 2, 3, 4).$$

The $p_\alpha$ must transform as components of a four-vector, since they are components of four-dimensional momentum. Furthermore, we know the transformation rule for a rela spinor $\psi$ under transformations of the Lorentz group.

Taking into account that, under arbitrary rotations and reflections of four-dimensional space, both sides of Eq. (2) must transform in like manner, we can determine to which of the fourth order real matrices considered in reference 7 the matrix $I$ corresponds. It is easy to see that here there are only two possibilities: The matrix $I$ must compare with $J$ or $-J$. Here, however, $\hbar$ must be considered not as a scalar (as has usually been assumed) but as a pseudoscalar, which changes sign upon four dimensional reflections but remains invariant for four-dimensional rotations. More exactly, this is the sole component of the matrix-pseudoscalar $T_4 = \hbar J$, and one must regard the numerical value of Planck's constant as the absolute value of the corresponding pseudoscalar.

The fact that the matrix $J$ should agree with $I$ rorrect as to sign leads to the imposition of a restriction on the choice of the remaining basic matrices considered in reference 7. Namely, they must correspond to the ones which were presented in the first or the second line of Table I from reference 7. For definitness,we shall assume that $J = I$ ( the choice of sign has no significance), so that

$$R^k = BS_k,\ R^4 = B,\ R = BI,\ J = I. \tag{3}$$

Thus, from the postulate of relativistic invariance, it follows that Eqs. (1) and (2) can be written in the form

$$-\hbar J (\partial\psi / \partial x^\alpha) = p_\alpha \psi, \tag{4}$$

where $\hbar$ is the component of a pseudoscalar. We note that the replacement of $\hbar$ by $-\hbar$ corresponds to a change of sign of $I = J$. It is well known that the results of non-relativistic quantum mechanics do not change if $i$ is everywhere changed to $-i$. Therefore they do not change if $\hbar$ is changed to $-\hbar$ in the basic formulas.

As the initial non-relativistic equation for the electron, to which we shall have to add the terms which make it relativistically invariant, we can take Pauli's equation, writing it in the form

$$\left[-i\hbar\frac{\partial}{\partial t} + \frac{1}{2m_0}\sum_{k=1}^{3}\left(-\hbar i\frac{\partial}{\partial x^k} + \frac{e}{c}A_k\right)^2 - e\varphi\right]\xi_0 + \frac{e\hbar}{2m_0 c}\sum_{k=1}^{3} H_k \sigma_k \xi_0 = 0, \tag{5}$$

where $e$ is the electronic charge and $A_k$ are the components of the vector potential. (See, for example, reference 9 etc. ).

From this equation (more exactly, two equations) it is necessary to go over to a real relation[1], since, on consideration of arbitrary transformations of the Lorentz group, the original entity, for which the law of transformation is defined, is a real spinor, and a spinor of first rank is not suitable for this purpose. Substituting for all the complex quantities in Eq. (5) the real quantities corresponding to them, we get

$$\left[-\hbar J\frac{\partial}{\partial t} + \frac{1}{2m_0}\sum_{k=1}^{3}\left(-\hbar J\frac{\partial}{\partial x^k} + \frac{e}{c}A_k\right)^2 - e\varphi\right]\psi_0 + \frac{e\hbar}{2m_0 c}\sum_{k=1}^{3} R_4 R^k H_k \psi_0 = 0. \tag{6}$$

In order to make Eq. (6) relativistically invariant, it is necessary to replace the operator

$$-\hbar J\frac{\partial}{\partial t} + eA_4 + \frac{1}{2m_0}\sum_k\left(-\hbar J\frac{\partial}{\partial x^k} + \frac{e}{c}A_k\right)^2,\quad A_4 = -\varphi$$

by

$$\frac{1}{2m_0}\left[-\left(-\hbar J\frac{\partial}{\partial x^4} + \frac{e}{c}A_4\right)^2 + \sum_k\left(-\hbar J\frac{\partial}{\partial x^k} + \frac{e}{c}A_k\right)^2 + m_0^2 c^2\right],$$

and $R_4 R^k H_k$ by $-J\,(RR^k H_k + R_4 R^k E_k) = -JF$. Thereby Eq. (6) goes over to

$$\left[-\left(-\hbar J\frac{\partial}{\partial x^4} + \frac{e}{c}A_4\right)^2 + \sum_k\left(-\hbar J\frac{\partial}{\partial x^k} + \frac{e}{c}A_k\right)^2 + m_0^2 c^2 - \frac{e\hbar}{c}JF\right]\psi = 0, \tag{7}$$

[9] D. I. Blokhintsev, *Principles of Quantum Mechanics*, GITTL, 1949

where $\psi$ is a real spinor. The Pauli equation (5) or (6) is obtained from Eq.(7) if we take

$\Psi = [\exp(-J\frac{m_0 c^2 t}{\hbar})]\psi_0$, disregard the quantities $\frac{m_0\hbar^2}{c^2}\frac{\partial^2\psi_0}{\partial t^2}$, $\frac{m_0 e^2\phi^2}{c^2}\psi_0$, $\frac{m_0 e\hbar}{c^2}\left(J\times\frac{\partial}{\partial t}\right)\phi + J\phi\frac{\partial}{\partial t}\psi_0$ and also discard the term $-\frac{e\hbar}{c}JR_4\underline{E}$.

The relativistic invariance of Eq. (7) comes out of the following : If one denotes the expression in the square brackets by the symbol $G$, then upon a similarity transformation characterized by the matrix $A_1$, $G$ goes over to $A_1 G A_1^{-1}$. Therefore the the equation $G\psi = 0$ goes over to $A_1 G A_1^{-1}(A_1\psi) = A_1 G\psi = 0$, whence it follows that it is actually relativistically invariant.

We rewrite Eq. (7) in the form

$$\left(\hat{E}^2 - c^2\sum_k \hat{P}_k^2 - m_0^2 c^4 + e\hbar cJF\right)\psi = 0, \qquad (8)$$

$$\hat{E} = \hbar J\frac{\partial}{\partial t} + e\varphi,$$

$$\hat{P}_k = -\hbar J\frac{\partial}{\partial x^k} + \frac{e}{c}A_k.$$

and this is the desired relativistically invariant equation which replaces the Dirac system of equations. (Strictly speaking, Eq. (8) is a system of four linear differential equations with real elements).

Already at first sight it is evident that the system of equations (8) is simpler than that which is obtained from Dirac's equations as a result of squaring. Actually,there are there four complex or eight feal functions, whereas our problem is reduced to the finding of four real functions-the components of a real spinor. Also, Pauli's equation is obtained from Eq. (8) considerably more easily than from Dirac's equations. The sense of Dirac's equations in the usual understanding is also different from that of Eq. (8). Inasmuch as the components of a real spinor are parameters which characterize a four-dimensional vector and an antisymmetrical tensor, Eq. (8) is, strictly speaking, a system of equations for the determination of four dimensional tensors. However, in the present article we shall not discuss in more detail the similarity and differences between Eq. (8) andDirac's equations. We also postpone the general analysis to later articles. In the current work, we restrict ourselves to the discussion of the electron in a hydrogen atom and to the calculation of the fine structure.

Before going over to the solution of this problem, we shall write Eq. (8) in a somewhat different form. Using Eq. (3) and reverting from real to complex quantities (assuming $JR_4 R^k \leftrightarrow i\sigma_k$, $\psi \leftrightarrow \xi$, etc.). we find that the fundamental relativistically invariant equation (8) for the electron can be written in the form

$$\left[\left(\hbar i\frac{\partial}{\partial t} + e\varphi\right)^2 - c^2\sum_k\left(-\hbar i\frac{\partial}{\partial x^k} + \frac{e}{c}A_k\right)^2 - m_0^2 c^4 + e\hbar c\,(i\sigma_k E_k - \sigma_k H_k)\right]\xi = 0. \qquad (9)$$

In such a form it has a particularly simple appearance and is convenient to use for solving concrete physical problems. However, for the consideration of questions connected with relativistic invariance, it is expedient to use a form of the type of Eqs. (7) or (8). The point is that under arbitrary transformations of the Lorentz group $\xi$ no longer is multiplied from the left by some matrix, in the way this occurred for three-dimensional rotations, but is transformed in a more complicated manner. Moreover, if we admit complex quantities, then we can no longer use the very simple transformation matrices (four dimensional tensors), and as a result, a demonetration of relativistic invariance becomes considerably complicated, etc.

## 3. EQUATIONS FOR THE ELECTRON IN A HYDROGEN ATOM IN THE ABSENCE OF EXTERNAL FIELDS

For the electron in a hydrogen atom, we can assume

$$A_k = 0, \quad \varphi = e/r. \qquad (10)$$

We will consider that the electron is in a stationary state, such that

$$\hbar i\frac{\partial}{\partial t}\xi = E\xi, \quad \xi = [\exp(-iEt/\hbar)]\xi_0, \qquad (11)$$

where $\xi_0$ does not depend on time. Therefore Eq. (9) takes the form

$$\left[\nabla^2 + \frac{1}{\hbar^2 c^2}\left[\left(E + \frac{e^2}{r}\right)^2 - m_0^2 c^4\right] + \frac{e^2}{\hbar c}i\left(\frac{x^k}{r^3}\right)\sigma_k\right]\xi = 0, \qquad (12)$$

whereupon in the following we will use the abbreviated notations:

$$\alpha = \frac{e^2}{\hbar c}, \quad \sigma_{(r)} = \frac{x^k\sigma_k}{r},$$

$$\nabla^2_{\vartheta\varphi} = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}.$$

Our problem consists of finding the energy levels $\mathcal{E} = E - m_0 c^2$ and the corresponding characteristic functions of Eq. (12). For its solution it is necessary to find a system of operators which commute with the operator

$$\nabla^2 + \frac{1}{\hbar^2 c^2}\left[\left(E + \frac{e^2}{r}\right)^2 - m_0^2 c^4\right] \qquad (13)$$

$$+ \frac{i\alpha\sigma_{(r)}}{r^2} = \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)$$

$$+ \frac{1}{\hbar^2 c^2}\left[\left(E + \frac{e^2}{r}\right)^2 - m_0^2 c^4\right] + \frac{\nabla^2_{\vartheta\varphi} + i\alpha\sigma_{(r)}}{r^2}$$

and with one another. Then the "proper" functions $\xi$ of equation (12) can be chosen in such a way that they will also be characteristic functions of all of these operators at the same time.

We can attain a separation of variables if we assume $\xi_0 = f(r)\begin{pmatrix} y_1(\vartheta,\varphi) \\ y_2(\vartheta,\varphi)\end{pmatrix}$ whence

$$-(\nabla^2_{\vartheta\varphi} + i\alpha\sigma_{(r)})\begin{pmatrix} y_1 \\ y_2\end{pmatrix} = \hat{L}y = \lambda y. \qquad (14)$$

Then the energy levels of the electron in a hydrogen atom will be determined from the equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{\hbar^2 c^2}\left[\left(E + \frac{e^2}{r}\right)^2 - m_0^2 c^4\right]\right. \qquad (15)$$

$$\left. - \frac{\lambda}{r^2}\right] f(r) = 0.$$

It is not difficult to verify that the operators (16) commute with (13) and $\hat{L}$.

$$\hat{M}_k = \hbar\left(-i\varepsilon^{kjs} x_j \frac{\partial}{\partial x^s} + \frac{1}{2}\sigma_k\right). \qquad (16)$$

This can be seen from the following: if we set $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, then

$$\left[\varepsilon^{kjs} x_j \frac{\partial}{\partial x^s}, \frac{x^t \sigma_t}{r}\right] = \frac{1}{r}\varepsilon^{kjs} x_j \sigma_s,$$

$$\left[\sigma_k, \frac{x^t \sigma_t}{r}\right] = \frac{2i}{r}\varepsilon^{kjs} x_j \sigma_s.$$

Furthermore, the operators $\hat{L}$, $\hat{M}_k$ and $\hat{M}^2 = \hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2$ commute with one another. Therefore we can choose $y_1$ and $y_2$ such that $y$, and consequently also $\xi_0$, will be a characteristic function of the operators $\hat{L}$, $\hat{M}_3$ and $\hat{M}^2$, simultaneously, i.e.,

$$M_3 y = \hbar m y, \qquad (17)$$

$$\hat{M}^2 y = \hbar^2 j(j+1) y, \qquad (18)$$

and the state of the electron in a hydrogen atom will be characterized by the quantum numbers $m, j$, $\lambda$ and the radial quantum number which we introduce below.

From the commutation relations between the $\hat{M}_k$, as is well known, it follows that for a given $j$, $m$ can take the values $m = -j, -j+1, \ldots, j-1, j$ (see references 6, 10, 11, etc.). It is possible to show, that from the stipulations (17) and (18) it follows that $y$ has the form:

$$y = \begin{pmatrix} y_1 \\ y_2\end{pmatrix} \qquad (19)$$

$$= \begin{pmatrix} (C_{j'} P_{j', m'} + C_{j'+1} P_{j'+1, m'})\, e^{im'\varphi} \\ \left(-\sqrt{\frac{j'-m'}{j'+m'+1}}\, C_{j'} P_{j', m'+1} + \sqrt{\frac{j'+m'+2}{j'-m'+1}}\, C_{j'+1} P_{j'+1, m'+1}\right) e^{i(m'+1)\varphi}\end{pmatrix},$$

where $C_{j'}$, $C_{j'+1}$ are arbitrary constants, $j' = j - \frac{1}{2}$ and $m' = m - \frac{1}{2}$ are integers,

$$P_{j', m'} = \sqrt{\frac{(2j'+1)}{2}\frac{(j'-m')!}{(j'+m')!}} \qquad (20)$$

$$\times\ (1-x^2)^{m'/2}\frac{1}{2^{j'} j'!}\frac{d^{j'+m'}}{dx^{j'+m'}}(x^2-1)^{j'},$$

$$x = \cos\vartheta.$$

We shall develop the proof of these assertions. The functions $y_1$ and $y_2$ can be expanded into a Laplace series, i.e., presented in the form

$$y_1 = \sum_{k,\,n} C_{kn} P_{n,k}\, e^{ik\varphi} = \sum_{k=-\infty}^{\infty} A_k e^{ik\varphi},$$

[10] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, GITTL, 1948

[11] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra*, Cambridge, 1935

$$y_2 = \sum_{k,\,n} D_{kn} P_{n,k}\, e^{ik\varphi} = \sum_{k=-\infty}^{\infty} B_k e^{ik\varphi}.$$

Substituting these expansions in Eq. (17), we get

$$y_1 = A_{m'} e^{im'\varphi}, \quad y_2 = B_{m'+1} e^{i(m'+1)\varphi}, \qquad (21)$$

$$m' = m - {}^1/_2$$

($m'$ is an integer). Here we can assume

$$A_{m'} = \sum_l C_l P_{l,m'}, \quad B_{m'+1} = \sum_l D_l P_{l,\,m'+1} \qquad (22)$$

($C_l$ and $D_l$ are constants). Further, using the the fact that

$$\sigma_1 \mp i\sigma_2 = \begin{pmatrix} 0 & 1\mp 1 \\ 1\pm 1 & 0 \end{pmatrix},$$

we obtain

$$\frac{1}{\hbar^2}\hat{M}^2 y = \left( -\nabla^2_{\vartheta,\varphi} + \frac{3}{4} + \begin{pmatrix} -i\frac{\partial}{\partial\varphi} & -i(\hat{l}_1 - i\hat{l}_2) \\ -i(\hat{l}_1 + i\hat{l}_2) & i\frac{\partial}{\partial\varphi} \end{pmatrix} \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (23)$$

$$= j(j+1)\,y, \quad \hat{l}_k = \varepsilon^{kjs} x_j \frac{\partial}{\partial x^s}.$$

Now use is made of the fact that

$$(l_1 \pm i\hat{l}_2) P_{l,\,m\mp 1/2} e^{i(m\mp 1/2)\varphi} \qquad (24)$$

$$= -i\sqrt{(l+m'+1)(l-m')}\, P_{l,\,m\pm 1/2} e^{i(m\pm 1/2)\varphi}, \quad m' = m - {}^1/_2$$

(See references 6, 10-12, etc. In this series of works, functions are used which are different from the $P_{l,m'}$ we use by the factor $(-1)^{m'}$. On using them, the sign before the right-hand side of (24) must be changed).

Substituting the expression (21) and (22) in (23) and taking (24) into account, we get

$$[l(l+1) + {}^3/_4 + m' - j(j+1)]\, C_l \qquad (25)$$

$$- \sqrt{(l+m'+1)(l-m')}\, D_l = 0 \quad (l \geqslant m'),$$

$$-\sqrt{(l+m'+1)(l-m')}\, C_l \qquad (26)$$

$$+ [l(l+1) - {}^1/_4 - m' - j(j+1)]\, D_l = 0 \quad (l \geqslant m'+1).$$

In order that Eqs. (25) and (26) have a non-zero solution, it is necessary that for $l \geq m' + 1$,

$$\begin{vmatrix} l(l+1) + {}^3/_4 + m' - j(j+1) & -\sqrt{(l+m'+1)(l-m')} \\ -\sqrt{(l+m'+1)(l-m')} & l(l+1) - {}^1/_4 - m' - j(j+1) \end{vmatrix} = 0 \qquad (27)$$

or

$$[l(l+1) - j(j+1) + {}^1/_4]^2 = (l + {}^1/_2)^2, \quad l = j \pm {}^1/_2. \qquad (28)$$

Thus, there are possible two solutions of Eqs. (25), (26), corresponding to the two possible choices of sign in Eq. (28), and the general solution will be a linear combination of these. We shall find it. If $l = j - \frac{1}{2} = j'$, then

$$D_{j'} = -\sqrt{\frac{j-m'}{j+m'+1}}\, C_{j'}, \qquad (29)$$

and if $l = j + \frac{1}{2} = j' + 1$, then

$$D_{j'+1} = \sqrt{\frac{j'+m'+2}{j'-m'+1}}\, C_{j'+1}, \qquad (30)$$

so that as the general solution, we actually get Eq. (19).

Now we can go over to the analysis of Eq. (14), from which must be found the third quantum number $\lambda$ and the ratio of $C_{j'+1}$ to $C_{j'}$.

Making use of the fact that $x_3 = r\cos\theta$, $x_1 \pm ix_2 = r\sin\theta\, e^{\pm i\phi}$, for the matrix $\sigma_{(r)} = \sum_k x^k \sigma_k / r$ we can write the expression

[12] H. A. Bethe, *Quantum Mechanics of the Simplest Systems*, ONTI, 1935

$$\sigma_{(r)} = \cos\vartheta\sigma_3 + \frac{1}{2}\sin\vartheta e^{i\varphi}(\sigma_1 - i\sigma_2) \quad (31)$$

$$+\frac{1}{2}\sin\vartheta e^{-i\varphi}(\sigma_1 + i\sigma_2) = \begin{pmatrix}\cos\vartheta & \sin\vartheta e^{-i\varphi}\\ \sin\vartheta e^{i\varphi} & -\cos\vartheta\end{pmatrix}.$$

Therefore, taking Eq. (19) into account, we can rewrite Eq. (14) in the form of a system of two equations:

$$[j'(j'+1)-\lambda]C_{j'}P_{j',m'} \quad (32)$$

$$+[(j'+1)(j'+2)-\lambda]C_{j'+1}P_{j'+1,m'}$$

$$-i\alpha\cos\vartheta C_{j'}P_{j',m'} - i\alpha\cos\vartheta C_{j'+1}P_{j'+1,m'}$$

$$+i\alpha\sqrt{\frac{j'-m'}{j'+m'+1}}\sin\vartheta C_{j'+1}P_{j',m'+1}$$

$$-i\alpha\sqrt{\frac{j'+m'+2}{j'-m'+1}}\sin\vartheta C_{j'+1}P_{j'+1,m'+1}=0;$$

$$-\sqrt{\frac{j'-m'}{j'+m'+1}}[j'(j'+1)-\lambda]C_{j'}P_{j',m'+1} \quad (33)$$

$$+\sqrt{\frac{j'+m'+2}{j'-m'+1}}[(j'+1)(j'+2)-\lambda]$$

$$C_{j'+1}P_{j'+1,m'+1} - i\alpha\left[\sin\vartheta P_{j',m'}\right.$$

$$\left.+\sqrt{\frac{j'-m'}{j'+m'+1}}\cos\vartheta P_{j',m'+1}\right]C_{j'}$$

$$-i\alpha\left[\sin\vartheta P_{j'+1,m'}\right.$$

$$\left.-\sqrt{\frac{j'+m'+2}{j'-m'+1}}\cos\vartheta P_{j'+1,m'+1}\right]C_{j'+1}=0.$$

Further, we use the fact that from the properties of $P_{j',m'}$ (reference 12, pp. 383-4 ) it is possible to derive the following relations:

$$-\cos\vartheta P_{j',m'} + \left(\sqrt{j'-m'}/\sqrt{j'+m'+1}\right)\sin\vartheta P_{j',m'+1} \quad (34)$$

$$= -\left(\sqrt{(j'-m'+1)(2j'+1)}/\sqrt{(j'+m'+1)(2j'+3)}\right)P_{j'+1,m'};$$

$$-\cos\vartheta P_{j'+1,m'} - \left(\sqrt{j'+m'+2}/\sqrt{j'-m'+1}\right)\sin\vartheta P_{j'+1,m'+1} \quad (35)$$

$$= -\left(\sqrt{(j'+m'+1)(2j'+3)}/\sqrt{(j'-m'+1)(2j'+1)}\right)P_{j',m'};$$

$$-\sin\vartheta P_{j',m'} - \left(\sqrt{j'-m'}/\sqrt{j'+m'+1}\right)\cos\vartheta P_{j',m'+1} \quad (36)$$

$$= -\left(\sqrt{(j'+m'+2)(2j'+1)}/\sqrt{(j'+m'+1)(2j'+3)}\right)P_{j'+1,m'+1};$$

$$-\sin\vartheta P_{j'+1,m'} + \left(\sqrt{j'+m'+2}/\sqrt{j'-m'+1}\right)\cos\vartheta P_{j'+1,m'+1} \quad (37)$$

$$= \left(\sqrt{(j'-m')(2j'+3)}/\sqrt{(j'-m'+1)(2j'+1)}\right)P_{j',m'+1}.$$

Taking into account these formulas and comparing the coefficients which occur with the $P_{j',m'}$, $P_{j'+1,m'}$ in Eq. (32), we obtain from Eq. (14)

$$[j'(j'+1)-\lambda]C_{j'} \quad (38)$$

$$= i\alpha\left(\sqrt{(j'+m'+1)(2j'+3)}\right.$$

$$\left.\sqrt{(j'-m'+1)(2j'+1)}\right)C_{j'+1},$$

$$[(j'+1)(j'+2)-\lambda]C_{j'+1} \quad (39)$$

$$= i\alpha\left(\sqrt{(j'-m'+1)(2j'+1)}\right.$$

$$\left.\sqrt{(j'+m'+1)(2j'+3)}\right)C_{j'}.$$

Equation (33) leads to the same relations.

From Eqs. (38) and (39) we obtain

$$C_{j'+1}/C_{j'} = \frac{[j'(j'+1)-\lambda]}{i\alpha\sqrt{\frac{(j'+m'+1)(2j'+3)}{(j'-m'+1)(2j'+1)}}} \quad (40)$$

$$= i\alpha\sqrt{\frac{(j'-m'+1)(2j'+1)}{(j'+m'+1)(2j'+3)}}$$

$$\frac{}{[(j'+1)(j'+2)-\lambda]},$$

whence

$$\lambda = (j'+1)^2 \mp \sqrt{(j'+1)^2-\alpha^2}. \quad (41)$$

We see that the quantum number $\lambda$ for given $j$ and $m$ can take only two values, corresponding to the two possible choices of sign in Eq. (41). If one neglects the quantity $\alpha^2$ and in Eq. (41) considers $\alpha^2 = 0$, then for this limiting case, instead of the quantum number $\lambda$ we can introduce another quantum number $l$, defined according to

$$\lambda = l(l+1), \quad l \geqslant 0. \tag{42}$$

This quantum number, for a given $j$ and $m$ can take the following two values:

$$l_1 = j' = j - 1/2, \quad l_2 = j' + 1 = j + 1/2. \tag{43}$$

Conditionally, we can consider that even in that case where we do not neglect the quantity $\alpha^2$, the state of the electron in a hydrogen atom is still characterized (aside from $j$ and $m$) by the quantum number $l$ (instead of $\lambda$), which can take the two values $l_1 = j - 1/2$ and $l_2 = j + 1/2$ (corresponding to the two possible values of $\lambda$: $\lambda_1$ and $\lambda_2$).

For given $\lambda$ (or correspondingly, $l$), Eq. (40) permits one to find the ratio of $C_{j'+1}$ to $C_{j'}$.

Now we can go over to the solution of the equation for the radial function (15), which we write down in the form:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - A + \frac{2B}{r} - \frac{C}{r^2}\right) f(r) = 0, \tag{44}$$

where

$$A = (m_0^2 c^2/\hbar^2)[1 - (1 + \mathcal{E}/m_0c^2)^2] \quad (\mathcal{E} = E - m_0c^2). \tag{45}$$

$$B = (m_0 e^2/\hbar^2)(1 + \mathcal{E}/m_0c^2), \tag{46}$$

$$C = \lambda - \alpha^2 = (j'+1)^2 \mp \sqrt{(j'+1)^2 - \alpha^2} - \alpha^2 = l'(l'+1), \quad l' > 0. \tag{47}$$

The solution of such an equation is very well known (see e.g. reference 13, pg. 438).
In particular, in order to obtain the energy levels $\mathcal{E}$, it is necessary to introduce the radial quantum number $k$ according to the formula *

$$B/\sqrt{A} = l' + k + 1 \tag{48}$$

(We are interested only in the discrete spectrum). The radial quantum number can take the values $k = 0, 1, 2, \ldots$. The energy levels are determined from the formula

$$1 + \mathcal{E}/m_0c^2 = [1 + \alpha^2/(l' + k + 1)^2]^{-1/2} \tag{49}$$
$$= [1 + \alpha^2/(k + 1/2 + \sqrt{1/4 + \lambda - \alpha^2})^2]^{-1/2}.$$

Expanding the right side of Eq. (49) into a power series in $\alpha^2$ and limiting ourselves to terms of the order of $\alpha^4$, we shall have

$$\mathcal{E}/h = -\frac{R}{n^2}\left(1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right), \tag{50}$$
$$n = j + 1 \mp \frac{1}{2} + k, \quad R = \frac{m_0 e^4}{4\pi\hbar^3} = \frac{\alpha^2 m_0 c^2}{2h}.$$

We have derived precisely the fine structure formula, which corresponds to that obtained upon solution of Dirac's system of equations.

Thus, both the relativistically invariant equation of Dirac and the relativistically invariant Eq. (8) introduced in the present work, similarly lead to the fine structure formula (50). The shift in the energy levels, which is corroborated by recently obtained experimental data [14], is not predicted in either case. But the approach to this question from the viewpoint of the validity of the (approximate) equation (8) must understandably be out of the ordinary. The question about which of the relativistically invariant equations, Eq. (8) or the Dirac equation, better reveals the characteristics of phenomena in the micro-world, remains open.

## 4. CONVERSION TO A SYSTEM OF DIFFERENTIAL EQUATIONS OF FIRST ORDER

The relativistically invariant Eq. (8) which we used is a differential equation (more exactly, a system of equations) of second order. Dirac's equations, however, are first order differential equations. The natural question arises as to whether it is possible also to derive Eq. (8) from a system of first order differential equations which are themselves relativistically invariant. We shall show this is actually possible.

Let us consider the system of first order differential equations which relate the components of the two real spinors $\psi_{(1)}$ and $\psi_{(2)}$:

* If we take the value of $l'$ negative, then the series for the radial function diverges.

[13] A. A. Sokolov and D. D. Ivanenko, *Quantum Field Theory*, GITTL, 1952

[14] Collection of Papers: *On Shift of the Levels of Atomic Electrons*", IIL, 1950

$$\nabla\psi_{(1)} - \frac{e}{\hbar c} JA\psi_{(1)}. \tag{51}$$

$$= \frac{m_0 c}{\hbar}\psi_{(2)} \quad \left(A = A_\beta R^\beta, \nabla = R^\beta \frac{\partial}{\partial x^\beta}\right),$$

$$\nabla\psi_{(2)} + \frac{e}{\hbar c} JA\psi_{(2)} = \frac{m_0 c}{\hbar}\psi_{(1)}. \tag{52}$$

These equations are relativistically invariant. In fact, since $e/\hbar c$ and $m_0 c/\hbar$ are pseudoscalars, then as a result of the symmetry transformation characterized by the matrix-four dimensional vector $A_1$, the first equation goes over to

$$-A_1 \nabla A_1^{-1} A_1\psi_{(1)}$$

$$-\left(-\frac{e}{\hbar c}\right) J(-A_1 A A_1^{-1}) A_1\psi_{(1)} = \left(-\frac{m_0 c}{\hbar}\right) A_1\psi_{(2)},$$

which corresponds to Eq. (51). Consequently, Eq. (51) is invariant with respect to transformations of the Lorentz group. Analogously, we convince ourselves of the relativistic invariance of Eq. (52).

A characteristic feature of Eqs. (51) and (52), as well as of Eq. (8), is that they do not change upon substitution of

$$\psi_{(1)} \text{ by } e^{Jf}\psi_{(1)}, \quad \psi_{(2)} \text{ by } e^{-Jf}\psi_{(2)}, \tag{53}$$

$$A_\alpha \text{ by } A_\alpha - \frac{\hbar c}{e}\frac{\partial f}{\partial x^\alpha}.$$

The conversion of all the quantities according to Eq. (53) can be considered as a gauge transformation. It is important only to note that the determination of the gauge transformation in accordance with Eq. (53) differs from the usual, in particular in that here $f$ must be considered as an arbitrary function multiplied by a pseudoscalar, which accordingly changes sign under four-dimensional reflections.

Let us multiply both sides of Eq. (51) from the left by the operator $\nabla + \frac{e}{\hbar c} JA$. Using formula Eq. (31) of reference 8, we obtain

$$\left[\nabla^2 + \frac{e}{\hbar c} JF + \frac{e}{\hbar c} JR^\alpha R^\beta A_\beta \frac{\partial}{\partial x^\alpha}\right.$$

$$\left. + \frac{e}{\hbar c} JR^\beta A_\beta R^\alpha \frac{\partial}{\partial x^\alpha} - \left(\frac{e}{\hbar c}\right)^2 A^2\right]\psi_{(1)} = \frac{m_0^2 c^2}{\hbar^2}\psi_{(1)}$$

or, considering that $R^\alpha A + AR^\alpha = 2\delta^\alpha_\beta 2A^\alpha$, we find

$$\left[\Box - \left(\frac{e}{\hbar c}\right)^2 A^2\right.$$

$$\left. + \frac{e}{\hbar c} J\left(F + A^\alpha \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} A^\alpha\right)\right]\psi_{(1)} = \frac{m_0^2 c^2}{\hbar^2}\psi_{(1)}.$$

Having assumed $\hat{E} = \hbar J \frac{\delta}{\delta t} + e\phi$, $\hat{P}_k = -\hbar J \frac{\delta}{\delta x^k} + \frac{e}{c} A_k$, we shall have

$$\left(\hat{E}^2 - c^2 \sum_k \hat{P}_k^2 - m_0^2 c^4 + e\hbar c JF\right)\psi_{(1)} = 0, \tag{54}$$

which agrees with Eq. (8).

Analogously, multiplying Eq. (52) from the left by the operator $\nabla - \frac{e}{\hbar c} JA$, we obtain

$$\left[\left(-\hbar J\frac{\partial}{\partial t} + e\varphi\right)^2 - c^2 \sum_k \left(\hbar J \frac{\partial}{\partial x^k} + \frac{e}{c} A_k\right)^2\right. \tag{55}$$

$$\left. - m_0^2 c^4 - e\hbar c JF\right]\psi_{(2)} = 0.$$

This equation differs from Eq. (8) only in that $J$ is replaced by $-J$, which corresponds to replacing $i$ by $-i$ in Eq. (9), but this can not change the physical results, so that Eq. (55) is equivalent to Eq. (54).

In this way, the relativistically invariant differential equation of second order which we studied, and which becomes directly a simple generalization of Pauli's equation (5), is obtained as a result of the system of first order Eqs. (51) and (52).

We note that for the consideration of discrete problems, another manner of writing Eqs. (51) and (52) may prove useful. Namely, we consider

$$\psi_{(1)} \leftrightarrow \xi = \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = \begin{pmatrix}\psi_{(1)1} + i\psi_{(1)3} \\ \psi_{(1)2} + i\psi_{(1)4}\end{pmatrix}, \tag{56}$$

$$\psi_{(2)} \leftrightarrow \eta = \begin{pmatrix}\psi_{(2)1} + i\psi_{(2)3} \\ \psi_{(2)2} + i\psi_{(2)4}\end{pmatrix}.$$

We use the fact that, according to reference 1,

$$-B\psi_{(1)} \leftrightarrow \begin{pmatrix}0 & -e^{-iu} \\ e^{-iu} & 0\end{pmatrix}\begin{pmatrix}\bar{\xi}_1 \\ \bar{\xi}_2\end{pmatrix} \tag{57}$$

$$= e^{-i(u+\pi/2)\sigma_2}\bar{\xi}.$$

For definiteness, we can assume, as in reference 1 (pg. 662), that $u + \pi/2 = 0$, since the parameter $u$ can be chosen arbitrarily. Therefore, from Eqs. (51) and (52), using Eq. (57), we get

$$\left[\sigma_k\left(\frac{\partial}{\partial x^k} + i\frac{e}{\hbar c}A_k\right) + \frac{1}{c}\left(\frac{\partial}{\partial t} - i\frac{e}{\hbar}\varphi\right)\right]\xi \tag{58}$$

$$= \frac{m_0 c}{\hbar}\sigma_2\bar{\eta},$$

$$\left[\sigma_k\left(\frac{\partial}{\partial x^k} - i\frac{e}{\hbar c}A_k\right) + \frac{1}{c}\left(\frac{\partial}{\partial t} + i\frac{e}{\hbar}\varphi\right)\right]\eta_1 \quad (59)$$

$$= \frac{m_0 c}{\hbar}\sigma_2\bar{\xi}.$$

In order to be able to study the system of equations (51), (52) in more detail, it is necessary to find out for which system of tensors the two real spinors $\psi_{(1)}$ and $\psi_{(2)}$ are parameters, and how the primary tensors are related among themselves. This question will be considered in the following paper.

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Translated by R. L. Eisner
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