potential energy by a potential function and the determination becomes incorrect. The limits of the $s, p, d, \ldots$, states can then be determined from Eq. (7) or Eq. (9), which reduce to the condition

$$[r^3\rho(r)]_{\text{макс}} = (1/24\pi^2)\,[4l(l+1)]^{3/2}.$$

Substituting $\rho(r)$ from the Thomas-Fermi-Dirac model (3), we get

$$Z_l = \gamma(Z)\,[4l(l+1)]^{3/2},\quad \gamma(Z) = \frac{1}{6\pi}\,[(x\psi)^{1/2} + \beta_0 x]^{-3}_{\text{макс}}. \tag{12}$$

By a numerical method, analogous to that of Ivanenko and Larin[9], we found for the limits of the s, p, d, and f states, $Z_l = 1, 4, 19, 53$, respectively, if $L = [l(l+1)]^{1/2}$, and $Z = 1, 4, 20, 55$ if $L = l + ½$. Thus the Thomas-Fermi-Dirac model gives $\gamma(Z)$ with $\gamma = 0.155$ (which follows from the simple Thomas-Fermi model) only for sufficiently large Z.

In conclusion, gratitude is expressed to Prof. D. D. Ivanenko and N. N. Kolesnikov for their consideration of the problems examined here.

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* In Born and Yang[7] the parameters of the density distribution of nucleons in the nucleus are determined, in essence, by the number of the "first appearance". In this case there correspond to the numbers of the first appearance of the p, d and f states in reference 7, l of one integer less, i.e., l 0, 1 and 2, respectively. The numbers of the first appearance of the g, h and i states under the same conditions do not agree with experiment.

** The relative difference in the expressions $l + ½$ and $[l(l+1)]^{½}$ is substantially greater for small l, since $l(l 1) = (l + ½)^2 - 1/4$.

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The Theory of the Dipole Lattice of Onsager

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In his paper[1] on phase transitions of the second kind in a plane dipole lattice, Onsager obtained the following expression for the logarithm of the partition function, per particle:

$$\ln \lambda^{(2)}(T) = \frac{1}{2\pi^2}\int_0^{\pi}\!\!\int_0^{\pi} \ln(\cosh 2\theta_1 \cosh 2\theta_2 - \sinh 2\theta_1 \cos\omega_1 - \sinh 2\theta_2 \cos\omega_2)\, d\omega_1\, d\omega_2. \tag{1}$$

Here $\theta_n = J_n/kT$ $(n = 1, 2)$; J_n is a constant characterizing the interaction between neighboring dipoles, and T is the temperature. Analysis[1,2] shows that Eq. (1) leads to a logarithmic divergence in the second derivative with respect to temperature, determined by the equation

$$\cosh 2\theta_1 \cosh 2\theta_2 - \sinh 2\theta_1 - \sinh 2\theta_2 = 0. \tag{2}$$

Taking one of the interaction constants to be zero, J_2, for example, Eq. (1) becomes a one dimensional integral:

$$\ln \lambda^{(1)}(T) = \frac{1}{2\pi}\int_0^{\pi} \ln(\cosh 2\theta - \sinh 2\theta \cos\omega)\, d\omega = \ln \cosh \theta, \tag{3}$$

which corresponds to a linear chain of dipoles.

It would seem natural[3] to conjecture, that for a three dimensional dipole lattice, $\ln \lambda^{(3)}(T)$ would become a triple integral:

$$\ln \lambda^{(3)}(T) = \frac{1}{2\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \ln(\cosh 2\theta_1 \cosh 2\theta_2 \cosh 2\theta_3 - \sinh 2\theta_1 \cos\omega_1 - \sinh 2\theta_2 \cos\omega_2 - \sinh 2\theta_3 \cos\omega_3)\, d\omega_1\, d\omega_2\, d\omega_3. \quad (4)$$

However, Eq. (4) is incorrect, because for certain values of the temperature, it leads to complex values for the free energy. In particular, for the simple case of an isotropic lattice $\theta_1 = \theta_2 = \theta_3 = \theta$, it may be easily verified that the function $\cosh^3 2\theta - 3 \sinh 2\theta$ has a negative minimum at $\theta = 1/4 \text{ Ar} \sinh 2$.

It turns out that even the following more general form for $\ln \lambda^{(3)}(T)$ will not give the desired result. We shall show that taking $\ln \lambda^{(3)}(T)$ to be the following triple integral

$$\ln \lambda^{(3)}(T) = \int_0^\pi\int_0^\pi\int_0^\pi \ln F(\theta_1, \theta_2, \theta_3; \omega_1, \omega_2, \omega_3)\, d\omega_1\, d\omega_2\, d\omega_3, \quad (5)$$

will not lead to a phase transition of the second kind if the function $F(\theta_k; \omega_k)$ $(k = 1, 2, 3)$ obeys the following conditions: 1) it is non-negative for all real values of its arguments; 2) $F(\theta_k; \omega_k)$ has no essential singularities.

The possible Curie points T_c are given by the zeros or poles of the function $F(\theta_k; \omega_k)$.* Let $F(\theta_k; \omega_k) = 0$ at $T = T_c$ and $\omega_k = \omega_k^0$. This point is at the same time a minimum of $F(\theta_k; \omega_k)$. In the neighborhood of this point F is a positive quadratic form in the variables $\tau = T - T_c$ and $x_k = \omega_k - \omega_k^0$, which by a rotation of the axes x_k, can be brought into the form:

$$F = a\tau^2 + \sum_{k=1}^{3} b_k x_k^2 + \tau \sum_{k=1}^{3} c_k x_k, \quad (6)$$

where a, b_k, c_k are functions of the structure constants.

The character of the singularity in $\ln \lambda(T)$ is determined from the integral of $\ln F$ over a small volume δ containing the point $x_k = 0$:

$$\mu(\tau) = \iiint_\delta \ln F(\tau, x_k)\, d^3x. \quad (7)$$

We transform variables according to the definition $y_k = \sqrt{b_k}\,(x_k + c_k/2b_k\, \tau)$. Considering τ to be sufficiently small, we integrate over a sphere of radius $r \gg (c_k/b_k)\,\tau$ with the center at the point $y_k = 0$. Changing to spherical coordinates, we obtain

$$\mu(\tau) = 4\pi \int_0^r \ln(a'\tau^2 + \rho^2)\,\rho^2\, d\rho \qquad \left(a' = a - \sum_{k=1}^{3} \frac{c_k^2}{4b_k^2} > 0\right) \quad (8)$$

After some simple manipulations we find that

$$\mu(\tau) = P(\tau) + A\tau^3 \arctan \frac{1}{\sqrt{a'}}\frac{r}{\tau}, \quad (9)$$

where $P(\tau)$ is an analytic function of τ and A is a coefficient. Eq. (9) shows that $\ln \lambda^{(3)}(T)$ together with its first two derivatives are continuous, while the third derivative has a finite jump.

Similar calculations show that the following expression for $\ln \lambda^{(n)}(T)$

$$\ln \lambda^{(n)}(T) = \int\int\cdots\int \ln F(\theta_k; \omega_k)\, d\omega_1\, d\omega_2 \ldots d\omega_n \quad (10)$$

$$(k = 1, 2, \ldots, n).$$

where the function F obeys the same conditions as before, gives a logarithmic divergence in the n'th derivative if n is even and a finite jump for odd n.

So far we have assumed that not all the second derivatives of F are zero at the point where $F(\theta_k; \omega_k) = 0$. Suppose now that all derivatives up to order $2s$ are zero. Then in the neighborhood of its zero F is a positive, homogeneous form of $2s$'th degree. We shall show that no divergences can occur before the third derivative of $\ln \lambda^{(3)}(T)$. The second derivative of $\mu(\tau)$ at $\tau = 0$ is

$$\left.\frac{d^2\mu}{d\tau^2}\right|_{\tau=0} = \iiint_\delta \left.\frac{d^2}{d\tau^2} \ln F(\tau; x_k)\right|_{\tau=0} d^3x \quad (11)$$

$$= \iiint_\delta \left.\frac{\left(\frac{\partial F}{\partial \tau}\right)^2 - F\frac{\partial^2 F}{\partial \tau^2}}{F^2}\right|_{\tau=0} d^3x.$$

The numerator and denominator are homogeneous polynomials in the variables x_k of degree $4s - 2$ and $4s$ respectively. Let us change to spherical coordinates ρ, ϑ, ϕ. Then $F^2(0; \omega_k) = \rho^{4s} F_1(0, \vartheta, \phi)$, where F_1 is the value of F on the unit sphere in the space of the x_k. Since the function F is positive and continuous on the unit sphere, $1/F_1^2$

is bounded. The numerator can be written in the form $\rho^{4s-2} F_2(\vartheta, \phi)$. Hence the integral (11) does not diverge, as claimed. A similar proof shows that in a space of any dimension n, all derivatives up to the $(n-1)$th, inclusive, are continuous.

The above suggests that the presence of a phase transition of the second kind in Onsager's plane lattice is connected with the dimensionality of the space. In particular, we may expect that Onsager's model does not give a phase transition of the second kind in a real three dimensional lattice and hence cannot explain the properties of a ferromagnet. This might indicate that in the three dimensional case not only interactions between neighboring dipoles need to be considered.

In conclusion, I would like to express my thanks to Prof. Iu. B. Rumer for his valuable suggestions and discussions during the course of this work.

Translated by R. Krotkov
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* In the case of a one dimensional lattice, the corresponding function $\cosh 2\theta - \sinh 2\theta \cos\omega$ has no zeros. Hence the theory does not give a phase transition in this case.

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The Spatial Distribution of Nuclear-active Particles in Broad Atmospheric Showers of Cosmic Rays

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IN autumn of 1952 we measured the spatial distribution of nuclear-active particles in broad atmospheric showers of cosmic rays at 3860 m of altitude. For this work we used a special arrangement with a considerable number of coincidence counters. This enables us to locate the axis and the number of charged particles of broad atmospheric showers which were of interest to us. The flux of nuclear- active particles was determined by the number of nuclear electron showers, produced in lead by the penetrating particles during the passage of a broad atmospheric shower.

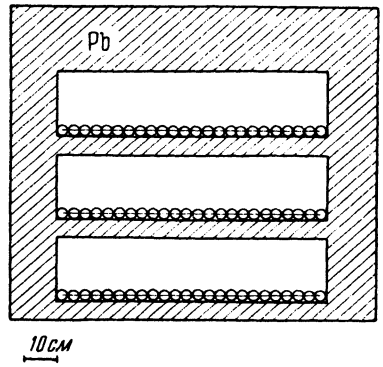

Fig. 1

The equipment for the observation of nuclear electron showers (Fig. 1) consisted of three groups of coincidence counters separated by shields of lead 6 cm thick. The presence of a 20 cm thick lead shield on the top of the counters allowed a reliable separation of penetrating particles belonging to the broad atmospheric shower from its photo-electronic component. The lead shielding on the bottom and the sides was 6 cm and 14 cm thick respectively.

The formation of the nuclear electron shower was characterized by the appearance of a discharge in two or more counters, placed in one row. The correction in lead due to μ mesons has been based on measurements of the number of δ showers produced in this apparatus by the hard component of the cosmic rays.

The relation between the observed number of nuclear electron showers produced by particles from broad atmospheric showers of a given energy at a given distance from the shower axis and the total number of broad atmospheric showers of same energy and same axis location can be expressed by the flux density of nuclear-active particles in the following way:

$$\frac{N_я}{N} = 1 - \exp\{-\rho\sigma(1 - e^{-x/\lambda})\}$$

Here ρ is the flux density of nuclear-active particles, σ is the counter surface recording the nuclear electron showers, x is the amount of matter in which the nuclear electron showers are produced, λ is the interaction path for nuclear-active particles ($\lambda = 160$ gm/cm^2 Pb). It is assumed that the probability of recording a nuclear electron shower originating in lead is unity, which may lower the flux density of nuclear-active particles*. The results of flux density measurements of nuclear-active particles obtained this way for different distances from the broad atmospheric shower axis