The Phenomenological Relations of Onsager

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IN Ref. 1, Popov established the fact that the phenomenological relations (in the thermodynamics of irreversible processes)

$$x_i' = \sum_{k=1}^{n} L_{ik} X_k,$$

$$i = 1, 2, \ldots, n \quad (x_i' = dx_i / dt)$$

are obtained directly as the first integrals of the set of differential equations

$$d^2 x_i / dt^2 = X_i \quad (i = 1, 2, \ldots, n), \tag{1}$$

which satisfy the conditions $\chi_i = \chi_i^0$ for $t = t_0$ and $\chi_i = 0$ for $t = +\infty$ $(i = 1, 2, \ldots, n)$, where

$$X_i = \partial(-\Delta S) / \partial x_i \tag{2}$$

and

$$\Delta S = -\tfrac{1}{2} \sum g_{ik} x_i x_k \quad (g_{ik} = g_{ki})$$

are positive definite quadratic forms.

The integrals of this system, which satisfy the above mentioned conditions, have the form

$$x_i'(t) = C_1 \eta_{i1} e^{r_1 t} + C_2 \eta_{i2} e^{r_2 t} + \ldots + C_n \eta_{in} e^{r_n t}$$

$$(C_k, \eta_{ik} \text{— are constants}),$$

where the r_i^2 are the positive roots of the algebraic equation of $2n$ th degree

$$\gamma(r) = \begin{vmatrix} g_{11} - r^2 & g_{12} & \ldots & g_{1n} \\ g_{21} & g_{22} - r^2 & \ldots & g_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ g_{n1} & g_{n2} & \ldots & g_{nn} - r^2 \end{vmatrix} = 0. \tag{3}$$

Eliminating $C_1 e^{r_1 t}, C_2 e^{r_2 t}, \ldots, C_n e^{r_n t}$ from the expressions for $\chi_i'(t)$ and $X_1, X_2, \ldots, X_n$, Popov* obtained

$$\begin{vmatrix} x_i' & \eta_{i1} & \eta_{i2} & \ldots & \eta_{in} \\ X_1 & \eta_{11} r_1 & \eta_{12} r_2 & \ldots & \eta_{1n} r_n \\ X_2 & \eta_{21} r_1 & \eta_{22} r_2 & \ldots & \eta_{2n} r_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ X_n & \eta_{n1} r_1 & \eta_{n2} r_2 & \ldots & \eta_{nn} r_n \end{vmatrix} = 0$$

$$(i = 1, 2, \ldots, n),$$

which can be written in the form

$$x_i' = \sum_{k=1}^{n} L_{ik} X_k \quad (i = 1, 2, \ldots, n),$$

where $L_{i1}, L_{i2}, \ldots, L_{in}$ $(i = 1, 2, \ldots, n)$ are functions of g_{ik} which do not depend on the constants of integration of $C_1, C_2, \ldots, C_n$.

With the help of an iteration process, which leads to the necessity of analyzing infinite series, Popov established the symmetry of the matrix

$$\begin{Vmatrix} L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & L_{22} & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & L_{nn} \end{Vmatrix}$$

For the particular case $n = 2$, he established this property by an algebraic method also.

Since the case $n = 3$ is very important for the application of the theory to specific problems, we demonstrate below the symmetry of the matrix in this case also.

For $n = 3$ the equation

$$\gamma(r) = \begin{vmatrix} g_{11} - r^2 & g_{12} & g_{13} \\ g_{21} & g_{22} - r^2 & g_{23} \\ g_{31} & g_{32} & g_{33} - r^2 \end{vmatrix} = 0$$

has three positive roots: r_1^2, r_2^2, r_3^2.

We assume that these roots are single and that*

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[1] K. A. Popov, J. Exper. Theoret. Phys. USSR **28**, 257 (1955); Soviet Phys. **1**, 336 (1955)

* $X_i = \dfrac{d^2 x_i}{dt^2} = \sum_{k=1}^{n} C_k \eta_{ik} r_k e^{r_k t} \quad (i = 1, 2, \ldots, n).$

* By means of a suitable transition to the limit, it is possible to show that these conditions are not restricting. Here $\gamma_{31}(0) = \begin{vmatrix} g_{12} & g_{12} \\ g_{22} & g_{23} \end{vmatrix}$, $\gamma_{32}(0) = -\begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix}$.

$$g_{32}\gamma_{31}(0) - g_{31}\gamma_{32}(0) \neq 0.$$

The integrals of the system (1) for stated conditions which characterize the physical processes, have the form

$$x_i = \sum_{\lambda=1}^{3} C_\lambda \gamma_{3i}(r_\lambda) e^{r_\lambda t}, \qquad (4)$$

where $\gamma_{3i}(r)$ is the algebraic complement of the corresponding element of the determinant $\gamma(r)$.

From this we obtain

$$X_i = \sum_{\lambda=1}^{3} C_\lambda \gamma_{3i}(r_\lambda) r_\lambda^2 e^{r_\lambda t} \quad (i = 1, 2, 3), \qquad (5)$$

$$\frac{dx_1}{dt} = \sum_{\lambda=1}^{3} C_\lambda \gamma_{31}(r_\lambda) r_\lambda e^{r_\lambda t};$$

$$X_i = \sum_{\lambda=1}^{3} C_\lambda \gamma_{3i}(r_\lambda) r_\lambda^2 e^{r_\lambda t} \quad (i = 1, 2, 3), \qquad (6)$$

$$\frac{dx_2}{dt} = \sum_{\lambda=1}^{3} C_\lambda \gamma_{32}(r_\lambda) r_\lambda e^{r_\lambda t}.$$

Eliminating $C_1 e^{r_1 t}$, $C_2 e^{r_2 t}$, $C_3 e^{r_3 t}$ from Eqs. (5) and (6), we obtain

$$\begin{vmatrix} X_1 & \gamma_{31}(r_1) & \gamma_{31}(r_2) & \gamma_{31}(r_3) \\ X_2 & \gamma_{32}(r_1) & \gamma_{32}(r_2) & \gamma_{32}(r_3) \\ X_3 & \gamma_{33}(r_1) & \gamma_{33}(r_2) & \gamma_{33}(r_3) \\ x_1' & \frac{1}{r_1}\gamma_{31}(r_1) & \frac{1}{r_2}\gamma_{31}(r_2) & \frac{1}{r_3}\gamma_{31}(r_3) \end{vmatrix} = 0;$$

$$\begin{vmatrix} X_1 & \gamma_{31}(r_1) & \gamma_{31}(r_2) & \gamma_{31}(r_3) \\ X_2 & \gamma_{32}(r_1) & \gamma_{32}(r_2) & \gamma_{32}(r_3) \\ X_3 & \gamma_{33}(r_1) & \gamma_{33}(r_2) & \gamma_{33}(r_3) \\ x_2' & \frac{1}{r_1}\gamma_{32}(r_1) & \frac{1}{r_2}\gamma_{32}(r_2) & \frac{1}{r_3}\gamma_{32}(r_3) \end{vmatrix} = 0;$$

so that

$$\Delta L_{12} = -\begin{vmatrix} \gamma_{31}(r_1) & \gamma_{31}(r_2) & \gamma_{31}(r_3) \\ \gamma_{33}(r_1) & \gamma_{33}(r_2) & \gamma_{33}(r_3) \\ \frac{1}{r_1}\gamma_{31}(r_1) & \frac{1}{r_2}\gamma_{31}(r_2) & \frac{1}{r_3}\gamma_{33}(r_3) \end{vmatrix}; \qquad (7)$$

$$\Delta L_{21} = \begin{vmatrix} \gamma_{32}(r_1) & \gamma_{32}(r_2) & \gamma_{32}(r_3) \\ \gamma_{33}(r_1) & \gamma_{33}(r_2) & \gamma_{33}(r_3) \\ \frac{1}{r_1}\gamma_{32}(r_1) & \frac{1}{r_2}\gamma_{32}(r_2) & \frac{1}{r_3}\gamma_{32}(r_3) \end{vmatrix};$$

$$\Delta = \begin{vmatrix} \gamma_{31}(r_1) & \gamma_{31}(r_2) & \gamma_{31}(r_3) \\ \gamma_{32}(r_1) & \gamma_{32}(r_2) & \gamma_{32}(r_3) \\ \gamma_{33}(r_1) & \gamma_{33}(r_2) & \gamma_{33}(r_3) \end{vmatrix},$$

where the determinant Δ, under our assumptions of single roots, is different from zero.

Consequently

$$\Delta(L_{21} - L_{12}) \qquad (8)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ F(r_1, r_2, r_3) & F(r_2, r_3, r_1) & F(r_3, r_1, r_2) \\ \frac{1}{r_1} & \frac{1}{r_2} & \frac{1}{r_3} \end{vmatrix},$$

where

$$F(u, v, w) = \gamma_{33}(u)[\gamma_{31}(v)\gamma_{31}(w) + \gamma_{32}(v)\gamma_{32}(w)].$$

The determinant on the right side of Eq. (8) is equal to zero. Consequently it is easy to show that

$$F(r_1, r_2, r_3) = [G_{33} - (g_{11} + g_{22})\rho_1 + \rho_1^2][(G_{31} + g_{31}\rho_2)(G_{31} + \rho_3 g_{31}) + (G_{32} + g_{32}\rho_2)(G_{32} + \rho_3 g_{32})]$$

is a symmetric function of the roots $\rho_1 (= r_1^2)$, $\rho_2 (= r_2^2)$, $\rho_3 (= r_3^2)$. Here G_{ik} denotes the algebraic complement of the element g_{ik} in the determinant obtained from $\gamma(r)$ for $r = 0$.

Making use of the elementary dependence of the roots on the coefficients of the characteristic equation $\gamma(r) = 0$, we have

$$[\gamma(r) \equiv \rho^3 - (g_{11} + g_{22} + g_{33})\rho^2 + (G_{11} + G_{22} + G_{33})\rho - g = 0],$$

$$\rho_1 + \rho_2 + \rho_3 = g_{11} + g_{22} + g_{33},$$

$$\rho_1(\rho_2 + \rho_3) + \rho_2\rho_3 = G_{11} + G_{22} + G_{33};$$

$$\rho_1\rho_2\rho_3 = g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix},$$

after some arrangements we obtain

$$F(r_1, r_2, r_3) = A + [G_{33}(g_{31}^2 + g_{32}^2) + (g_{11} + g_{22})(g_{31}G_{31} + g_{32}G_{32}) + G_{31}^2 + G_{32}^2]\rho_2\rho_3 - [(g_{31}G_{31} + g_{32}G_{32})(b + G_{33}) - g_{33}(G_{31}^2 + G_{32}^2) - g(g_{31}^2 + g_{32}^2)]\rho_1,$$

where A does not depend on ρ_1, ρ_2, ρ_3 and $b = G_{11} + G_{22} + G_{33}$.

Direct evaluation of the coefficients ρ_2, ρ_3 and ρ_1 in the expression $F(r_1, r_2, r_3)$ shows that they are equal to zero and that

$$F(r_1, r_2, r_3) = F(r_2, r_3, r_1) = F(r_3, r_1, r_2) = A.$$

Consequently, the determinant on the right side of Eq. (8) is equal to zero, and since $\Delta \neq 0$, $L_{12} = L_{21}$. In a similar fashion it can be shown that $L_{13} = L_{31}$.

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On the Theory of the Hall and Nernst-Ettinghausen Effects in Semiconductors with Mixed Conductivities

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The Hall and the Nernst-Ettinghausen voltages and the distribution of current carrier concentrations are calculated for a semiconductor with mixed conductivities located in a non-homogeneous magnetic field. Recombination of current carriers and energy levels due to impurities are taken into account.

1. INTRODUCTION

SOME papers[1-3] have recently been published in which the effect of body and surface recombinations on the Hall effect in semiconductors has been investigated. The basic assumption in all these works has been Ohm's law in a form valid only for semiconductors without impurities, a form in which the mean free path of the current carriers does not depend on their velocity.

This work investigates both the Hall effect and the thermomagnetic Nernst-Ettinghausen effect under the assumption of a more general dependence of the length l of the mean free path of the current carriers on the velocity v :

$$l = \Phi(T)\, v^n, \tag{1}$$

where $\Phi(T)$ is some function of the temperature T, and n is any given number. It will be shown that such a more general dependence of l on v leads to substantial quantitative changes in both the current carrier concentrations and the voltages in both effects, as well as to qualitative changes in the case of the Nernst-Ettinghausen effect. By analyzing the general case of a non-homogeneous magnetic field it is possible to determine how the non-uniformity of the magnetic field near the edges of pole-pieces affect the phenomena under consideration.

The following assumptions are made:

1. The magnitude of the exponent n in Eq. (1) has the same value for both electrons and holes. This is equivalent to assuming an identical scattering process for both electrons and holes by the phonons.
2. The primary current (either electric or thermal) is directed along the x axis, and the magnetic field, which is a function of y, is directed along the z axis ($H = H_z$). Therefore the electric fields of both the Hall and the Nernst-Ettinghausen effects are functions of y only.

2. BASIC EQUATIONS

The generalized differential equation of Ohm's law for electrons and holes, as derived by Tolpygo[4], can be written:

$$\mathbf{j}_+ = -eu_+N_+\Big\{-\mathbf{E} + \frac{kT}{e} \tag{2a}$$
$$\times\left[\nabla \ln N_+ + \frac{n+1}{2}\nabla \ln T\right] + \frac{a_n u_+}{c}$$
$$\times\ [-\mathbf{E} + \frac{kT}{e}(\nabla \ln N_+ + \frac{n+1}{2}\nabla \ln T)] \times \mathbf{H}\Big\};$$

$$\mathbf{j}_- = -eu_-N_-\Big\{-\mathbf{E} \tag{2b}$$
$$-\frac{kT}{e}\left[\nabla \ln N_- + \frac{n+1}{2}\nabla \ln T\right]$$

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